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ABSTRACT

This collection of papers was solicited from a variety of educators in an effort to provide direction to teachers, curriculum developers and textbook publishers on topics in mathematics education that are consistently mentioned in recent national reports and study groups. The consensus of reports sponsored by various professional associations suggests the topics addressed in this document. It contains papers on: (1) mathematics content; (2) geometry at the middle school level; (3) probability and statistics at the middle school level; (4) the inclusion of number theory in the mathematics curriculum; (5) estimation and mental computation; (6) problem solving; (7) mental images and visual thinking; (8) the role of technology in mathematics education; and (9) the development of teachers' conceptualization of mathematics. Because many of the professional groups addressing these topics have had difficulty in establishing definitions and directions for teachers, this document is an effort to address each topic as it would (should or could) appear in textbooks for students moving from arithmetic to algebra. (TW)

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A CHANGE IN EMPHASIS

(Mathematics in the Transition Years, Grades 7 & 8)

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A CHANGE IN EMPHASIS

(Mathematics for the Transition Years, Grades 7 & 8)

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April, 1986

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INTRODUCTION

These papers were solicited from the listed educators in an effort to give direction to teachers, curriculum developers and textbook publishers on those topics consistently mentioned in recent, prestigious national reports and study groups. The consensus of reports from such groups as the American Association for the Advancement of Science (AAAS); The College Entrance Examination Board (CEEB); the Conference Board of Mathematical Sciences (CBMS); the National Assessment of Educational Progress (NAEP); the National Council of Supervisors of Mathematics (NCSM); the National Council of Teachers of Mathematics (NCTM); the National Science Foundation (NSF); and others, is that we need a change of emphasis in the content and delivery of mathematics. The following topics that are consistently stressed in these reports.

estimation/mental computation
problem solving
probability/statistics
geometry/measurement
spatial thinking
computers/calculators
number theory

Although several of these professional groups mentioned above have established some definitions and indicated direction for teachers, the effort to date has been fragmented by efforts with supplemental materials, and suggested schemes for curriculum enhancement. This document is an effort to address each topic as it would (could)(should) appear in textbooks for students moving from arithmetic to algebra. The "experts" contributing to this document are easily identified as knowledgeable in their respective topics by their previous study, research and experiences.

1) What mathematics should be in this transition period? It is easy to talk about teaching units on a particular topic, but by injecting topics into a traditional program, something must give. To keep the effort honest we must give special attention to the total mathematics content that we should be teaching youngsters in this transition between arithmetic and algebra. This concern is addressed by Lynn Steen, President of Mathematical Association of America (MAA) and Professor of Mathematics at St. Olaf College.

2) Informal and concrete geometry have been consistently recommended for mathematics content for seventh and eighth grade mathematics, but it is debatable about our progress in curriculum development in this topic. Mary Montgomery Lindquist, Professor of Mathematics Education at Columbus College, is co-author for a K-6 curriculum program whose pedagogy is measurement, and is the editor for the NCTM yearbook on geometry (1987). She documents her position that at least forty percent of the seventh and eighth grade curriculum should be geometry and measurement.

3) Every major mathematics curriculum pronouncement since 1975 has emphasized the importance of probability and statistics in the middle school years, especially in grades 7 and 8. Albert Shulte, mathematics curriculum supervisor for the Oakland Schools in Pontiac, Michigan, co-author of textbooks and editor for the NCTM Yearbook on Probability (1981) presents the position that students can learn appropriate material and teachers find it exciting to teach. He recommends seven content categories for seventh and eighth grade students in his paper.

4) The development of solid mathematics curriculum which satisfies both student and content needs must maintain students' skills and understandings, and provide students with opportunities to develop problem solving skills related to applications of mathematics. One topic in mathematics that accomplishes this according to John A. Dossey, Professor of Mathematics/Mathematics Education at Illinois State University, and President of NCTM, is number theory. He presents his position on the topics of number theory that should be included in seventh and eighth grade mathematics.

5) Few topics in the mentioned list have been slighted in curriculum textbooks as much as estimation/mental computation. Like its partner "problem-solving" the topic is vaguely defined in a developmental sequence, requiring higher level cognitive skills and must be developed over a long period of time. Such material defies present administrative and curricula management schemes. Addressing this topic are Barbara and Robert Reys, authors, researchers and teachers at the University of Missouri-Columbia.

6) With pushes from the NCSM and NCTM, problem solving is becoming an integral part of the school mathematics curriculum, according to Jessie Rudnick and Steven Krulik, Professors of Mathematics Education at Temple University and co-authors of various books and materials on problem solving. However, they argue that we are teaching students to solve problems and not to be problem solvers.

7) Most mathematics curricula stress routines, automated practice and sequential reasoning. The recommendations from recent national reports conflict with such emphasis, and encourage applications of

mathematics both within the content and to worldly situations. Many important ideas require mental images, and visual thinking plays an important role in mathematical problem solving. This spatial component to problem solving is presented by Grayson H. Wheatley, Professor of Mathematics Education and Director of the School Mathematics and Science Center at Purdue University.

8) How does technology impact upon the mathematics curriculum and how do we make appropriate use of computer technology are questions addressed by Bob Kansky, Professor of Mathematics Education at Texas A & M University and member of NCTM steering committee for "The Impact of Computing Technology on School Mathematics". Kansky discusses various categories of software tools for 7th and 8th grade mathematics in his position paper.

9) Changing the emphasis of content as addressed by the first eight position papers in this document is crucial, but a most important variable in affecting change is the teachers' conceptualization of mathematics. The belief that most teachers' concept of mathematics emphasizes computation and the ability to produce correct answers quickly, is a premise for the paper by Thomas L. Good, Professor of Behavioral Psychology at the University of Missouri-Columbia and researcher on teacher expectations and student performance. He argues that teachers need to pay more attention to the developmental part of the lesson and to concentrate on helping students understand mathematics. His position paper is the glue that holds the others together and binds it to the content specification outlined in the first paper by Lynn Steen.

These position papers were made possible by the Monsanto Fund as part of a joint curriculum development project with the Parkway School District and the University of Missouri-Columbia.

During the 1985-86 school year a staff of three teachers and a curriculum supervisor are developing, teaching and revising lessons that are attentive to the advice in this document. The Parkway School District is located in St. Louis County, Missouri serving 22,000 students (K-12). The Instructional Services Center is located at 12657 Fee Fee Road, Creve Coeur, Missouri 63146.

The partnership: John Mason, Director of Monsanto Fund, Monsanto Corporation
Michael Silver, Assistant Superintendent of Curriculum and Instruction, Parkway Schools
Douglas Grouws, Professor of Mathematics Education University of Missouri-Columbia Principal consultant for the project.

The project proposer and director is Richard Lodholz, mathematics coordinator for the Parkway School District.
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A TIME OF TRANSITION: MATHEMATICS FOR THE MIDDLE GRADES

Lynn Arthur Steen

Seventh and eighth grades occupy a paradoxical position in the school curriculum. On the one hand, these grades serve as a holding pattern for students in transition from elementary school to high school, a place where those who are behind can catch up and where others can review basic skills to insure appropriate preparation for the challenge of high school. On the other hand, student experiences in these grades provide a crucial filter that first begins to track students into career options, where some drop away from academic programs in general, and even more drop away from significant work in mathematics and science. For better or for worse, seventh and eighth grades form the gateways to high school study and for life-long career options.

Mathematics is of central importance in the transitional role played by the middle school curriculum. Since mathematics is one of the crucial academic competencies, success or failure in grades seven and eight can have significant impact on each student's future: what happens here really counts for the rest of a student's life.

The major objective of mathematics in the middle grades is to provide a transition from arithmetic to algebra, and in so doing to provide both a holding pattern for entrance to high school mathematics and a critical filter for subsequent careers. In addition, for the majority of students, it is the beginning of the end of their study of mathematics. Appalling as it may seem, it is nonetheless true that what the average U.S. citizen knows about mathematics is pretty well defined by the mathematics

curriculum of the seventh and eighth grades.

Hence the curriculum of the seventh and eighth grades can be viewed from three perspectives: as a transition from arithmetic to algebra; as a critical filter to track students for subsequent high school study; and as a major component of general mathematical education for the majority of citizens.

From Arithmetic to Algebra

The Second International Assessment of Mathematics (14) identified four tracks of eighth grade mathematics in the United States: two thirds of the students are in the regular track, with the remaining third split roughly equally between remedial classes, enriched classes, and early algebra classes. One of the striking findings in this study is that each of these four classes ended the year less able than the next higher group had begun the year. In other words, the approximately four-year span in mathematical achievement of students who began eighth grade expanded to a seven-year span by the time they left eighth grade.

The reality is, therefore, that some students enter high school having finished elementary algebra, whereas others remain so mired in arithmetic that they will probably not learn any algebra before they drop out of required mathematics. The holding pattern of seventh and eighth grade is in reality a dispersal pattern, launching some into solid high school work in mathematics and science while grounding others in school and career patterns that will prevent students from ever achieving success in quantitatively based professional fields.

Another disturbing finding from the Assessment is that only one class in twenty at the eighth grade level makes regular use of calculators. (The data from the Assessment study was gathered in 1981-82, so perhaps current practice may be a bit improved.) It has been at least ten years since NCTM

recommended that calculators should be used regularly from grades five and on, or once children had learned the basic arithmetic skills (see [12], for example). Nevertheless, despite the ubiquitous role of calculators in society, school practice still emphasizes repetition of old-fashioned arithmetic skills rather than appropriate use of arithmetic in dealing with quantitative problems.

Mastery of arithmetic should not be the ultimate objective of elementary school mathematics. It should, rather, be a means to a more important end: the ability to solve quantitative problems. Yet the International Assessment documents what has been asserted in many of the national critiques of school education--that today's students are less able than were their predecessors to solve complex problems, although they are just as able to do basic arithmetic. In achieving its major purpose, the "back-to-basics" movement helped strengthen basic skills--the means--at the expense of problem-solving skills--the ends.

Critical Filter

Much has been written (e.g., [8]) about the critical role of middle school mathematics in closing the door--especially for minority students and young women--to the possibility of scientific or managerial careers. Recent data suggests that girls at least are continuing the study of mathematics in high school in about the same proportion as are boys, although minority students still are filtered out in disproportionate numbers.

Students have varied talents, and society needs to help place students in different positions according to their talents. Students with talent in mathematics should be encouraged to pursue studies leading to careers in which advanced mathematics is required. Students for whom mathematics is a

real struggle should be helped early to develop talents in other areas that will not be continually frustrated by lack of mathematical achievement. The mathematics curriculum in grades seven and eight is the first of a succession of filters in this necessary and important aspect of American education.

However, for the filter to work well it must reflect in an appropriate manner the actual mathematical skills needed for future scientific and business careers. Moreover, it must give students the benefit of the doubt, erring on the side of encouragement rather than discouragement. That suggests, at a minimum, that the curriculum in these grades must be broad rather than narrow, opening up the diversity of mathematical approaches to problems rather than emphasizing mastery of a narrow set of skills. Students should encounter measurement, arithmetic, algorithms, calculators, geometry, data, programming, statistics, estimation, conjecture, reasoning. All these and more should be visible in the curriculum, to provide a variety of soils in which the fragile seeds of mathematical aptitude may germinate.

Quantitative Literacy

Popular concern for science and mathematics training in the schools centers on two issues: the quality of preparatory work for those who will become future scientists and engineers, and the extent of public literacy in scientific, technological and quantitative issues that have come to dominate so much of practical life and public policy. For the half of our population that does not go on to college, the substance of seventh and eighth grade mathematics is for all practical purposes the educational capstone in quantitative literacy.

Thomas Jefferson referred to the "educated citizenry" as the only proper foundation of democracy. In the first 150 years of our country, the

necessary mathematical component in that public literacy was primarily the arithmetical skills conveyed by the common schools--the "'rithmetic" of the three R's. But the rise of technology as a dominant force in society changes fundamentally the kinds of knowledge appropriate to public literacy. The fundamental reason for urging change in the school mathematics curriculum is the rapid change now occurring in the role of mathematics in society [6].

Many observers have tried in different ways to capture the essence of these new requirements, to articulate a replacement for old-fashioned "rithmetic." In its 1980 Agenda for Action [12], the National Council of Teachers of Mathematics called for a focus on "problem solving" as the appropriate emphasis in school mathematics. More recently, the Commission on Behavioral and Social Sciences and Education of the National Research Council has called for emphasis on "active teaching and learning of reasoning" [11]. And in a 1981 occasional paper of the Council for Basic Education [15], Steve Willoughby argues that "thinking is the basic skill to be taught in school mathematics."

Although various observers do not quite agree on what to call the therapy, they all agree on the diagnosis: singular emphasis on arithmetic is inappropriate as the major objective of middle school mathematics. In the early grades, arithmetic is appropriate as a central focus of the curriculum. But in the upper grades, especially in grades seven and eight, that emphasis quickly becomes excessive, counterproductive, and inimical to sound educational goals.

Recommendations

This analysis of the different roles played by school mathematics in grades seven and eight leads to a pleasant and perhaps surprising

conclusion: the requirements of a sound curriculum are the same for each of these three objectives. Although the goals of transition, filtering, and literacy are fundamentally different, curricula appropriate to meeting these objectives are fundamentally the same. It is entirely possible, and now more urgent that ever, to move towards these three objectives with a single transformed curriculum.

That curriculum must stress problem solving, reasoning, and thinking--not calculation. It must provide an introduction to the diversity of mathematics and to the range of its applications, not just to the traditional "word problems" of arithmetic. It must encourage students to extend their natural curiosity and playfulness to quantitative and geometrical ideas, to give them confidence in their own abilities and resourcefulness. And above all else it must not give the impression that mathematical and quantitative ideas are the product of authority or wizardry. Students must come to "own" their own mathematical abilities just as they own (and therefore invent) their own music, sport, and world views.

Here are some examples of components from which such a curriculum could be built. The list is not in any way complete, but serves instead to illustrate the type of flexible life-serving skills that should be emphasized in seventh and eighth grade mathematics:

-Number sense. Newspapers and business reports are full of gigantic numbers, without any human experience as correlative support. Students need to be able to deal with such large numbers, and to make reasonable estimates by extrapolating from common experience. [Example: How many feet tall is the World Trade Center? How many people have ever lived on earth? How do U.S. expenditures on defense compare with total expenditures on education?]

-Understanding units. All scientific quantities that enter daily lives arrive with units of measurement without which the numerical quantities are totally meaningless. Students need to know about different units of measurement--how to use them, where to look them up. [Example: How much more costly is electricity than natural gas? What does the "ppb" mean in environmental news stories about environmental hazards?]

-Interpreting Surveys. The world we live in is dominated by opinion polls and clinical trials. Students need to understand via compelling examples how to interpret these reports--the importance of random samples, the use of controls, the misinterpretations of correlation. [Example: How does one judge competing claims for new cancer treatments?]

-Estimation. Numbers enter our lives most often from external sources--calculator answers, news reports, sales promotions. Students need to be taught that by using simple strategies of estimation they can easily determine--with a few seconds thought--if figures are reasonable. [Example: What is the effect on total loan payments of a change from 8% to 16% in an automobile loan? How much total sales tax does each student pay each year?]

-Understanding Risk. The mathematics of probability is at the core of many private decisions and public debates, from lotteries to smoking, from hazardous dumps to nuclear power. Students need to be introduced quite early--and seventh and eighth grade is not too early--to the subtleties involved in comparing risks, to learn both from experience and from mathematics how to build a better common sense. [Example: How does one compare the risks from flying with those of driving? What can one conclude about accident rates from the price of automobile insurance?]

-Common Geometry. Everyday life is full of complex geometric situations,

from estimating the cost of housing to building a deck. Students need experience in applying the simple geometry of rectangles and triangles to realistic situations. [Example: How much wallpaper is required for a room? How much roof overhang is appropriate for maximum solar efficiency?]

These types of problems will lead naturally to open-ended discussions in which there are many suitable strategies of solution. Inventive, involved children can develop a variety of approaches, exercising both their intellect and their imagination. Appropriate teaching strategies would emphasize group investigation, oral reports, written analyses--other basic competencies that should be reinforced in the mathematics classroom. Students would begin to see mathematics as a human activity with room for individuality, creativity, and argument. This would leave them well prepared for the rigors of high school mathematics, provide legitimate clues to future career choices, and send into society citizens who are not afraid of quantitative reasoning in daily life.

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GEOMETRY AND MEASUREMENT

Mary Montgomery Lindquist

All major curriculum committees over the past 100 years, with the notable exceptions of the Committee of Fifteen and the Cambridge Conference, have recommended the inclusion of informal or concrete geometry in Grades 7 and 8 (Peterson, 1973). Many of the recommendations captured the flavor of geometry proposed in this paper and there is some indication that the recommendations were heeded, at least in some textbooks (Breslich, 1925, Strayer & Upton, 1929).

Almost 40 years ago, Maurice Hartung presented a paper entitled "Improving the Mathematics Programs in Junior High Grades," to the Second Annual Conference on Arithmetic. The first improvement that he noted was "work in informal geometry has been considerably extended in scope and modified in type" (Hartung 1947, p.49). This treatment of geometry was characterized as follows:

> (1) The pupil is helped to formalize certain geometric concepts that he already has and also to acquire informally a few new ones. Thus most pupils who reach Grade VII already have some ideas about circles, parallel and perpendicular lines, angles, squares, and certain other geometric configurations. These primitive notations are made more precise, and the proper terminology is attached to them. For example, pupils have had a good deal of experience with congruent figures but only rarely any experience with the term "congruence." (2) The pupil learns certain important relationships,... . For example, he learns the relation of equality of angles to equality of sides, as in the isosceles and equilateral triangles, and the proposition that the sum of the angles is 180 degrees. (3) The pupil perfects his skill in drawing, particularly through the using of simple instruments, such as the protractor, and by making constructions with the compass, including simple designs. (4) The pupil increases his ability in measurement, both direct and indirect, and learns how to calculate, by formulas, the perimeters, areas, and volumes of the simpler closed geometric form. (Hartung 1947, p.55)

Although Hartung saw these characteristics as improvements he had some concerns. There were not enough realistic measurement problems with actual measuring involved, there was a lack of variety in the construction tools used, there was insufficient background before formulas were presented, and there was no unifying idea such as function. Although he had these concerns, he felt that great progress had been made and, therefore, suggestions were harder to make for improvement of geometry than for arithmetic.

I wonder what he would think today. I am not as optimistic about the progress we have made in the last 40 years as he appeared to be then. I do agree that suggestions are harder to make for geometry than for arithmetic, but probably for a different reason. It is more difficult not because there is less need for change, but because the structure of geometry is illusive. We have come to believe in a logical ordering of the arithmetic curriculum, but have little consensus on the structure of the geometric curriculum.

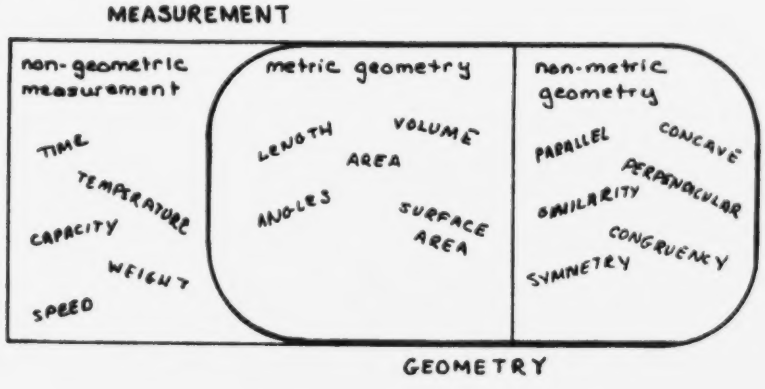
In this paper, I will outline some underlying structures that may assist curriculum developers in making decisions about geometry and measurement. Then, I will recommend some topics appropriate for junior high school or pre-algebra students, that is for students after the sixth grade and before algebra. I do this with trepidation, because I waver between but am nearer to the statement, "I do not care what geometry you teach, but how you teach it" than to "I really want a student to know how to do _____ and to know what a _____ is." After suggesting content topics appropriate for pre-algebra students, I will give some suggestions for the organization of the geometry.

Structural Considerations

Some people think of measurement as a part of geometry while others

think of the two topics separately. It is probably more helpful to think of them as shown in Figure 1. Thus, there are three areas to consider: non-geometric measurement, non-metric geometry, and metric geometry. We can glean from research and writings about measurement, ideas about attributes in the measurement set including both non-geometric measurement and geometric measurement. Likewise, we can look at proposed structures for geometry when considering attributes in the non-metric geometry as well as the metric geometry. Those attributes in the intersection must be considered as both geometry and measurement.

Figure 1. Relationship between measurement and geometry.



The measurement process can help structure the approach to teaching measurement. Consider the following outline of steps in the measuring process (Reys, Suydam & Lindquist, 1984):

I. Identifying the attribute by comparing objects
   A. Perceptually
   B. Directly
   C. Indirectly

II. Choosing a unit
   A. Arbitrary
   B. Standard

III. Comparing the unit to the object

IV. Finding the number of units
   A. By counting
   B. Using instruments
   C. Using formulas

V. Reporting the number of units (p. 143)

If measurement is considered in this light, there are natural levels through which children proceed. Keeping these in mind could help eliminate premature use of instruments and rote learning of formulas.

Measurement Level 0: The student compares objects perceptually, directly, and indirectly.

Measurement Level 1: The student chooses a unit and compares the object to the unit in order to find the number of units through counting or by using simple instruments. At this level the student would also estimate the number of units or the measurement and use simple instruments to construct given measurements.

Measurement Level 2: The student uses formulas or non-simple instruments to find the measurement. The student examines precision and error of measurement.

The work of the van Hieles can provide structure to the approach to geometry. A concise summary of the first three of their levels was given by Burger and Shaughnessy (1986):

Level 0 (Visualization): The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 1 (Analysis): The student reasons about geometric concepts by means of informal analysis of component parts and attributes. Necessary properties are established.

Level 2 (Abstraction): The student logically orders the properties of

concepts, forms abstract definitions and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

It may also be helpful in structuring the geometry to consider an attribute by process matrix. (See Figures 2 and 3) These matrices, which can be modified to include other attributes, serve as a check and balance to the geometric portion of the curriculum. If a curriculum is only describing figures in terms of names and definitions, then it is not providing a well-rounded curriculum.

Both the measurement levels and the van Hiele levels give guidance to when and how different processes and attributes can be considered. For example, areas of two figures could be compared directly very early in the curriculum, but areas of two non-similar parallelograms would not be compared abstractly until students were at least at Level 1 of each of the levels.

It is assumed that the curriculum prior to the pre-algebra would provide experiences on Levels 0 and 1 of both the measurement and geometry levels. If this is not the case, then the curriculum in pre-algebra needs to be expanded.

Content

As mentioned in the introduction, the way the geometry is approached is probably as important as the particular content. However, there are seven topics that should be reserved for these grades and become the focus of the pre-algebra program. The treatment of these topics should be on Level 1 and approaching Level 2 of the van Hiele levels and on Level 2 of the measurement levels.

1) Measurement systems. Metric and the conventional measurements each should be examined as a system. In addition, the process of measurement should be explored as students are asked, for example, to describe how

GEOMETRY - TWO-DIMENSIONAL

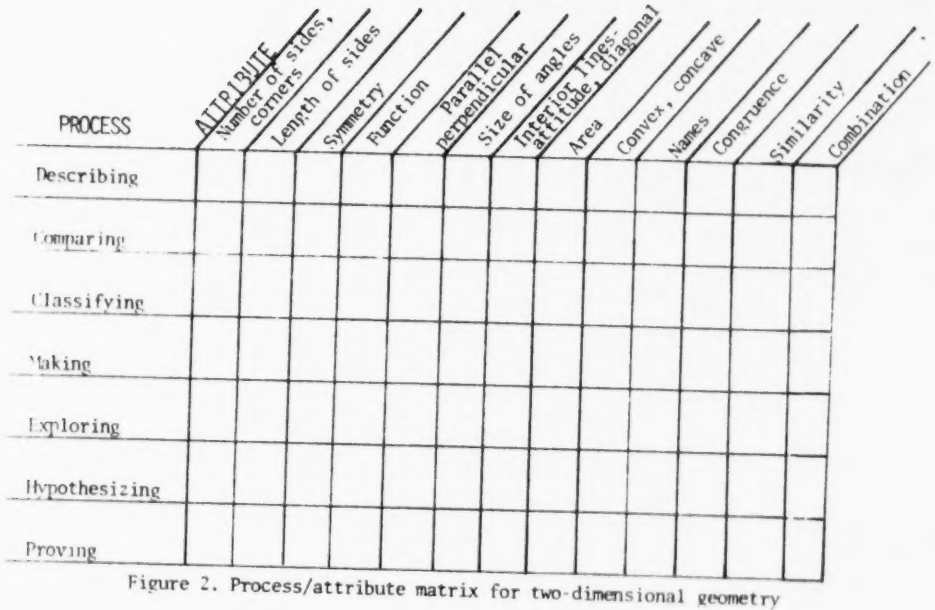


Figure 2. Process/attribute matrix for two-dimensional geometry

GEOMETRY - THREE-DIMENSIONAL

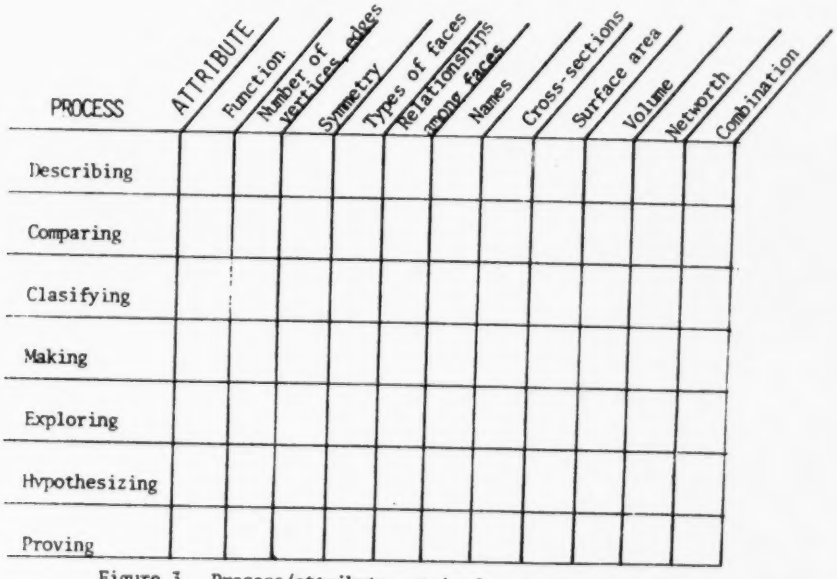


Figure 3. Process/attribute matrix for three-dimensional geometry

measuring length is similar and different to measuring area. Certainly, applications of measurement (both geometric and non-geometric) should be included.

2) Formulas. Formulas for area, volume, and surface area should be derived as short cuts for counting. The formulas should be related, whenever possible, according to classification schemes. How are all the surface area formulas for prisms related? How are the area formulas for rectangles, squares, parallelograms, and trapezoids related? In addition, areas and volumes of other shapes should be investigated and other problems posed such as find the area of a rhombus if the lengths of the diagonals are known. The Pythagorean formula, Pick's formula, and other relationships could be investigated. Applications of formulas in real life situations in which approximation is necessary should be included.

3) Classification schemes. Examining classification schemes should serve to pull together much information about two-dimensional and three-dimensional figures. For example, triangles can be classified according to their angles or the lengths of their sides. What differences occur in the sorting of triangles if these two different schemes are used? How does the class of equilateral triangles relate to the class of isosceles triangles? How do quadrilaterals fit a classification of concave and convex figures? What figures fit the classification: two diagonals of a quadrilateral are perpendicular. One type of figure, such as pyramid could be investigated by examining all the possible attributes of three-dimensional figures.

4) Similarity and congruence. Emphasis here should be on application as well as investigating conditions that assure similarity or congruence of two figures. Under what conditions are two quadrilaterals similar? two triangles congruent?

5) Angle relations. Relationships among angles of given figures

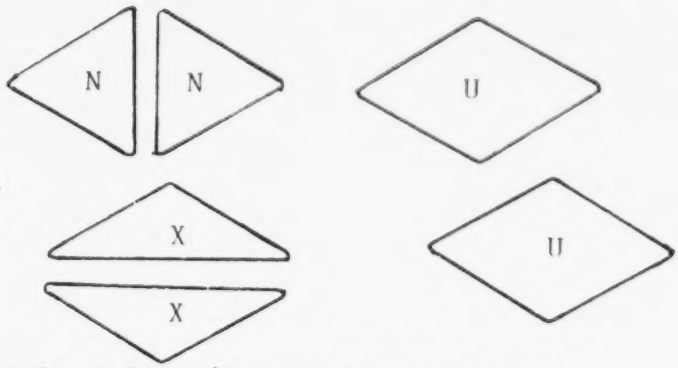
(triangles, parallel lines, angles in circles) should be explored. These provide for many suitable problems for pre-algebra by integrating problem solving situations. For example, if the base angles of a triangle are each twice as large as the other angle, what are the sizes of the angles?

6) Coordinate systems. Many figures can be investigated using a coordinate system. Equations of lines, transformations, both those that preserve congruence and those that preserve similarity, distance formulas all lend themselves to appropriate activities. The use of different coordinate systems could be explored.

7) Mini-deductive sequences. Simple deductive sequences should be explored and represented. For example, if given the three figures below, one can deduce the following:

In area: $2N = 1U$
$2X = 1U$

Therefore, $N = X$



Organization

The curriculum should devote at least 40 percent of its time to geometry and measurements. This should not be done in isolation from the other topics, but this amount of time devoted to these topics should be reached by integrating geometry and measurement into many other topics.

There should be a unit on measurement, both non-geometric and geometric, early in the program. This would focus on content topic 1 and some of the area formulas in content topic 2. Then, these concepts and skills should be used throughout the remainder of the program whenever possible. For example, when studying graphing, the area of a rectangle with

a given perimeter could be included. The list of ways to integrate measurement is endless, and without this constant use there is little reason to expect students to develop measurement skills and concepts.

After studying classifications in geometry with heavy emphasis on three- dimensional figures, the remainder of the formulas could be developed. I see this as a second unit which could include some of the other content topics. However, I mainly see the other topics included when appropriate. For example, two or three lessons could be developed on mini-deductive systems or angle relations when variables are being studied. The coordinate system could be investigated when studying graphing.

In addition, there should be mini-units developed that are independent of the two units mentioned and the lessons included with other topics. These should be designed to be used at any time. These would provide the flexibility to increase the program to two years or to challenge those in a one-year program. These units should be investigative in nature, allowing for the discovery of many theorems, experimenting with constructions of all types, exploring number and other patterns in geometry, and solving geometric problems not presented in the other parts of the program.

In all of the activities, there is a need for physical materials to support the program. Certainly, at a minimum, there needs to be measurement instruments (rulers, tapes, protractors) and constructing instruments and materials (compasses, three dimensional materials, geoboards or other physical materials). Technology should be utilized and many of the mini-units could be geared to computers.

In summary, this should be a time to pull together many ideas that have (hopefully) previously been introduced and to formalize some of the vocabulary and relationships. On the other hand, it is a time to explore

new ideas-- those that will be useful in further study of geometry, those that are challenging and interesting for these students, and those that will be useful in everyday living.

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STATISTICS AND PROBABILITY IN THE PRE-ALGEBRA SEQUENCE (GRADES 7 AND 8)

Albert P. Shulte

I. The Case for Statistics and Probability

The case for including statistics and probability as standard topics in the school mathematics curriculum has been made many times in recent years.

A major reason for teaching statistics and probability is the connection with, and preparation for, daily life. Newspapers, magazines, TV programs, reports from organizations--all these sources and many more present us with data in the form of graphs, tables, probabilities of occurrence, ratios of success and failure, margins of error on polls. We deal with batting averages, probability of rain, frequency of different types of cancer, likelihood of a particular candidate winning, our chances of injury when traveling in different ways (cars vs. airplanes, for example), whether the economy is improving or declining, and countless other applications.

Every major mathematics curriculum pronouncement for many years has stressed the importance of teaching statistics and probability. The NCSM Position Paper on Basic Mathematical Skills listed "Using mathematics to predict" and "Reading, interpreting, and constructing tables, charts, and graphs" as two of its ten basic mathematical skills (NCSM, 1977). The Agenda for Action (NCTM, 1980, p. 7) indicated the need for data collection and interpretation. The NACOME Report recommended "that instructional units dealing with statistical ideas be fitted throughout the elementary and secondary curriculum" (CBMS, 1975, p. 197). The CBMS Report "Mathematics in Grades K-12: What is Still Fundamental and What Is Not"

(CBMS, 1983) also made a strong statement in favor of including statistics and probability throughout the curriculum.

The work of the Quantitative Literacy Project of the American Statistical Association has addressed two major problems in teaching statistics and probability: (1) development of a significant body of material designed for teacher use; (2) training a significant cadre of teachers to use the materials developed. This project has demonstrated that appropriate materials can be learned by students from the upper elementary school through the senior high school, and that teachers find the materials exciting to students and to themselves.

II. What Content Should be Included in Grades 7 and 8 in the Pre-Algebra Sequence?

A. Basic Ideas of Probability

This should include the following topics, presented in interesting contexts:

Empirical (experimental) probabilities--probabilities determined as a result of doing experiments; theoretical probabilities--probabilities determined by thinking mathematically about the situation (these may use sample spaces, tree diagrams, or listings of events); doing experiments after determining theoretical probabilities to see that the results bear out the theoretical work). Both with experimental and theoretical probabilities, students should start with equally likely cases and then proceed to situations where the probabilities are not equally likely. Expressing probabilities as numbers between 0 and 1, inclusive. Impossible probabilities are shown to have probability 0, and certain probabilities are shown to have probability 1. Complementary probabilities (if the probability of event A is p, the probability of event A', where A does not occur, is 1-p). Probabilities of compound events (for example, tossing

three coins), both in the equally-likely and non-equally-likely situations.

Independent events; mutually exclusive events. Computing the probability of A and B both occuring; computing the probability of A or B occuring. A simple introduction to conditional probability. This can be done by looking at a situation such as drawing marbles, and examining the situation step by step. The formal notations of reduced sample space and the term "conditional probability" don't have to be introduced--just the ideas.

B. Sampling, Variability and Randomness

Sampling to learn about characteristics of a population (e.g., the preferences of certain TV shows) is a natural follow-up to probability. This should include discussion of various sampling schemes, and the need for random sampling. Students should see that even if a cluster sample or stratified sample is taken, that the cluster or the items sampled within each stratum are determined randomly. Samples dealing with the same characteristic should be drawn several times, to show students the variability in sampling, and also the greater accuracy resulting from pooling sample results.

C. Simulations

The power of simulations should be introduced early and continued throughout a student's statistical study. Simulation is especially useful when carrying out the actual activity is either extremely costly or extremely time-consuming, or when mathematical techniques do not exist for getting the desired information.

Simulations can be used both in game situations and as simulations of real or pseudo-real situations. At some point in the simulation process, the use of random digit tables and, later, the computer should be

introduced.

D. Descriptive Statistics

Students should learn the standard measures of central tendency (mean, median, mode), measures of position (quartiles, percentiles) and measures of spread (range, inter-quartile range). The standard deviation is not necessary at this level--it should be deferred to the high school curriculum.

Students should learn some of the techniques from Exploratory Data Analysis (line plots, stem-and-leaf plots, box plots) as simple ways to organize and display data. They should learn to make interpretations based on examination of these plots. Students should be exposed to bivariate data (plotting height against weight, for example). They should make scatter plots and learn about positive and negative associations as shown in the scatter plots. The term "correlation" need not be introduced at this time, but the idea may certainly be conveyed. Students can draw lines of best fit to scatter plots, using medians, and can use these lines for predictions.

E. Inferences

Students in grades 7 and 8 should have some practice in making inferences based on the data they have collected or been presented with. These inferences can include such experiences as: whether a game or device appears to be fair or biased; making decisions about a population based on a sample (and revising these decisions based on repeated or accumulated samples); predicting the occurrence of future events based on data.

F. Surveys

Students should conduct surveys to gain answers to questions. They could do this at any time after studying sampling techniques and sampling variability (section B). The teacher should stress the designing

of questions that avoid bias, and the necessity for each potential participant in the survey to have an equal chance of being selected.

G. Graphing

Some of the Exploratory Data Analysis techniques mentioned above (line plots, stem-and-leaf plots, box plots) are graphic techniques, as are scatter plots. In addition to these techniques, students should learn to use traditional statistical graphs (bar graphs, line graphs, circle graphs, pictographs). When studying these graphing techniques, they should learn to use proper techniques of scaling and learn to avoid deceptive graphing techniques.

III. Organization in the Course

Statistics and probability should be incorporated in the seventh and eighth grade pre-algebra curriculum as chapters. It is desirable to work through a substantial block of material, sequentially developed, before gaining enough familiarity with the topics and their underlying concepts to be able to apply them in new situations. Once the topics have been studied in some depth, they should be used from that point on, through periodic review and through systematically incorporating them in the assignments as other topics are studied.

If the materials are to be spread through two years of study, I would suggest the following:

Grade 7--A chapter on Basic Ideas of Probability early in the year. This chapter would take about three weeks of teaching time.

Work with sampling, variability and randomness could constitute a second chapter later in the year. This also provides a natural review for the probability. This chapter could then conclude with some work on surveys.

The third chapter in grade 7 could be a chapter on simulations.

This applies and reinforces the probability and the work on randomness. Grade 8--A chapter on descriptive statistics through box plots, early in the year. There should be a lot of practice organizing data sets using the Exploratory Data Analysis techniques.

A short chapter on bivariate data could be offered later in the year. This could be followed by a chapter on conventional statistical graphing.

A final chapter in the eighth grade could deal with inferences, as described in IIE.

It is extremely important that once the statistical ideas are developed, they are used from then on. Only in that way will these topics become tools which the students will use easily and naturally.

IV. Assumptions About K-6 Mathematics

There do not need to be any changes in the mathematical content of the K-6 mathematics curriculum to accommodate this statistical/probabilistic content in grades 7 and 8. However, these topics would benefit by the inclusion of statistical and probabilistic topics in the K-6 curriculum.

Elementary school students should have had some experience with probability. This can be introduced through game situations.

Some time should be given in the elementary school to the classroom collection and examination of data. This would not only benefit later study of statistics and probability, but would help prepare students for laboratory science work, as well.

Elementary students should have some exposure to sampling. This can again be done largely through games, if desirable.

The elementary school curriculum should have a well-designed sequence of graphing activities. These should follow the development, beginning in

the primary school, from object graphs (using the actual objects being considered) through picture graphs (using pictures or sketches of the objects) to symbolic graphs. (See Shulte, 1981, pp. 33-40.) If this is carried out well, students could enter the seventh and eighth grades with considerable experience with bar graphs, line graphs, pictographs, and circle graphs. This would mean that the eighth grade chapter suggested in the previous section could focus on more sophisticated forms of these graphs, which would be welcomed in science and social studies classes. (Both science and social studies tend to use graphs of greater complexity than are taught presently in comparable grades in mathematics.)

V. Closing Comments

Students studying the amount and type of statistics and probability suggested in this paper would enter the algebra program "statistically literate." They would be able to use statistics and probability in their further work in mathematics, science and social studies. They would also have the appropriate foundation for a more extensive study of statistics and probability at the junior-senior level in high school. Most importantly, they would be able to apply principles of statistical and probabilistic reasoning to everyday-life situations.

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NUMBER THEORY TO JUNIOR HIGH SCHOOL MATHEMATICS

John A. Dossey

The identification, selection, and sequencing of material for the junior high school mathematics curriculum are tasks which need urgent national attention in mathematics and mathematics education. The development of a solid mathematics curriculum which satisfies both student and content needs at this level will have to be able to maintain students' skills and conceptual understandings, while providing avenues for strengthening the conceptual bases upon which these skills and concepts are constructed. In addition, the topics selected should allow for the integration of students' knowledge about different branches of mathematics. Finally, students should have ample opportunities to develop problem solving skills related to modern applications of mathematics. One area of mathematics, currently found in the curriculum, which satisfies these criteria is number theory.

Number Theory

The study of the properties of the positive integers and the application of these properties to other sets of numbers is a topic which can provide activities relevant to the stated goals for junior high school mathematics. But more importantly, it provides junior high school students with an opportunity to develop mathematically while they are dealing with new material.

The topics, and their sequence, which seem appropriate for the junior high school grades are the following:

Divisibility and Types of Numbers

-odd and even

-the divides relation as a special case of the Division Algorithm

-divisible by n (for $2 < n < 9$)

-prime and composite

-Sieve of Erathosthenes

Divisors and Multiples

-Fundamental Theorem of Arithmetic

-greatest common divisor gcd (a,b)

-least common multiple lcm [a,b]

-Basic property ab = gcd (a,b.) x lcm [a,b]

-Euclidean Algorithm

Special Projects

-Number of Factors of a Natural Number

-Perfect Numbers

-Abundant and Deficient Numbers

-Sum of Factors of a Natural Number

-Distribution of Primes

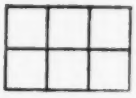
While several of these topics are currently found in the junior high school mathematics curriculum, they need a refocus on both purpose and presentation. At present, the number theory topics appear because they are needed to handle some of the mechanics of rational number arithmetic. These uses appear in finding the sum of two fractional numbers with unlike denominators or in finding the "simplest" representative for a given rational number.

While these roles might remain, number theory should be studied as an area of mathematics in its own right. It provides models for later work in algebra and tools for discovery and problem solving in a wide variety of

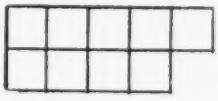
settings. Students can easily follow algorithm development in number theory because of their familiarity with the whole numbers and their properties. However, more is needed than just changing the focus on the reasons for including number theoretic topics in the junior high school curriculum. These topics must be included in the curriculum in a fashion which allows students to think visually about numbers and their properties, while at the same time learning to deal with algorithmic processes.

Divisibility and Types of Numbers

In the development of the topics of divisibility and types of numbers, students should see even and odd numbers as numbers which can be represented as rectangles of dimensions 2 x n or as such rectangles plus a single unit.



$6 = 2 \times 3$



$9 = 2 \times 4 + 1$

From this viewpoint, it is easy to develop properties of even and odd numbers and generalize these properties to all integers. From this point, it is also easy to develop the standard forms for even and odd numbers in terms of the more standard 2n and 2n + 1 representations.

The divides relationship can then be thought of as generalizing this relationship from building rectangles having dimensions 2 x n (divisibility by 2) to building rectangles of dimension 3 x n (divisibility by 3), 4 x n (divisibility by 4), ...

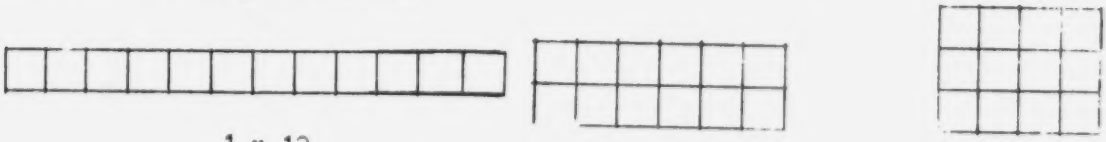
The corresponding patterns can then be captured in number arrays (as to the right) which feature divisibility by 4 in the fourth column. In addition, such an

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22...		

array prepares for future study of patterns of prime numbers in the Sieve of Erathosthenes and work with congruence classes.

The columns of the array can also be used to show the relationship between the divides relation, a case of divisibility with remainder zero, and the application of the division algorithm where the remainder is a non-negative integer less than the divisor. The first column contains the numbers having remainder 1 when divided by 4, the second column contains the numbers having remainder 2, etc.

Prime numbers can be characterized as numbers whose only representations are as rectangles of dimension 1 x p for some whole number $p > 1$. Composite numbers are then numbers having at least two different rectangular representations, for example 12 can be represented by three different rectangular arrays.



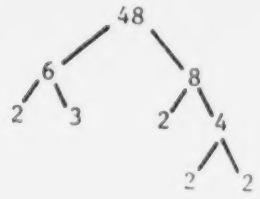
1 x 12 2 x 6 3 x

The next natural question to ask is how many different representations are possible for a given positive integer. The student is then ready to consider the question of the number of factors a given positive integer possesses. This can lead to a special student project.

Divisors and Multiples

The preceding topics provide a solid foundation for looking at the topics of greatest common divisor and least common multiple. Following the standard development of the prime factorization of a given positive integer using factor trees and then more efficient algorithms, such as that shown below, students should have an opportunity to try and develop methods of using the prime factorizations to find the greatest common

divisor and least common multiple of a pair of positive integers.



$48 = 2 \times 2 \times 2 \times 2 \times 3$
$= 2^4 \times 3$

3	1
2	3
2	6
2	12
2	24
2	48

Concrete models can be used to assist students in finding these values. For example, the prime factorization of 28 ($2^2 \times 7$) and 42 ($2 \times 3 \times 7$) can be modeled with two different types of cards with prime numbers written on them. The greatest common divisor asks for the largest number which can be written from the prime factors of either of the numbers. This can be easily seen to be the number formed by the product of the primes appearing in both sets.

28 = [2] x [2] x [7]
42 = [2] x [3] x [7]

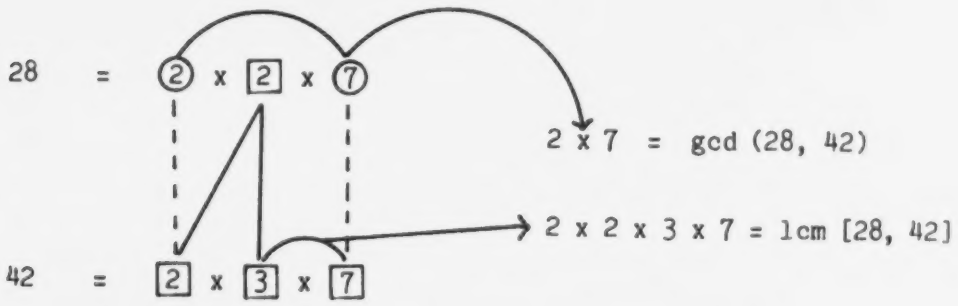
gcd (28, 42) = 2 x 7

The least common multiple is the smallest number one can build from the set of prime factors of the numbers, but from which you can find the factors necessary to write either of the numbers. In this case, such a number would have to have at least 2 factors of 2 in order to get 28 and this would satisfy the requirement for factors of 2. To write the 42 from the new factor set, we would need at least one factor of 3. To write either number, we need one factor of 7. Hence, it appears that the smallest factor set meeting the requirement is 2 x 2 x 3 x 7.
The resulting product is 84, the least common multiple of 28 and 42.

These approaches can be easily extended into the more common and formal symbolic approaches to the calculation of the gcd and lcm through finding conditions on the exponents of the primes appearing in the prime

factorizations of the numbers involved.

A side product of the card approach to determining these numerical quantities is the development of the property that for two natural numbers a and b, we have ab = gcd (a,b) x lcm [a,b]. This appears as shown in the illustration for 28 and 42.



The combination of the factors of the gcd and the lcm result in the same factor set as the product of the two numbers 28 and 42. This result gives a rapid way of finding the gcd or lcm if you know one of the numbers and the product of the two numbers, especially if one is working with a hand calculator.

This process can be improved greatly with the introduction of the Euclidean Algorithm for finding as illustrated in Maxfield and Maxfield (1972) for finding the greatest common divisor for a pair of positive integers. The use of this algorithm to find the gcd (74844,137592) quickly shows that it is far more efficient than the approach using the prime factorization of the numbers involved. In addition to the question of efficiency, the introduction of the Euclidean algorithm provides students opportunities to examine two different methods of solving the problem of finding the gcd of two integers. Both algorithms have their strong points. The first is much easier to understand and the second is deeper and much more efficient in operation.

Special Projects

Once students have the basic concepts and skills of number theory in hand, they have a rich vista of possible projects ahead of them. Students with a flair for counting and figuring the number of possible ways things happen can be steered toward finding out about the number of positive integer factors a given positive integer possesses. This can be accomplished through experimentation, collecting data, and looking at patterns in the exponents in the number's prime factorization.

Other students, having an interest in the history of mathematics can be steered toward finding out about the number theory of the Greeks. Topics of special interest here are perfect, abundant, and deficient numbers and their properties. Additional efforts might be directed toward describing the number theory efforts of Pythagoras and Euclid.

Other students, especially those interested in computing, might be encouraged to report on the infinitude of primes, the largest known prime, and the distribution of prime numbers. The reference list contains several references which will assist students in these searches.

Summary

The study of topics from number theory, while spread across the span of the junior high school program, will provide students with an opportunity to review earlier skills while extending them, assist in student understanding of the relation of divides, create opportunities to examine the development of algorithms and compare their efficiency. These are critical goals to any mathematics program of study, but they are especially needed in today's junior high school curriculum.

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SOME THOUGHTS ON MENTAL COMPUTATION AND ESTIMATION FOR THE JUNIOR HIGH

Barbara J. Reys and Robert E. Reys

There are many important elements of a quality junior high school mathematics program. Few topics have traditionally been overlooked to a greater extent than mental computation and estimation. Close relatives of problem solving, these topics are more difficult to teach than more standard topics because high level thinking is required of both processes.

"Is that answer reasonable?" is a question we often ask students, yet one few students understand. It calls for reflection and a gut level sense as well as a complete understanding of the problem. It calls for a brief period of concentrated reflection. How can students be taught this important component of the problem solving process?

Mental computation brings with it a different set of obstacles. As junior high teachers, we shudder to think of our students writing a problem such as 300 x 50 in a traditional vertical format then methodically recording zeros to obtain the product of 15,000. The obstacle here is again that of resistance to thinking. Most of us have experienced this phenomenon ourselves wanting to release our focus on the bigger problem, therefore mechanically carrying out the application of the traditional algorithm. In a sense, letting our hands do the work so our mind can be free to travel in other directions. This non-application of mental computation can be justified. However, we sense that with students, the problem is a different one. Many students have developed a mindset about mathematics--one that has been nurtured all through their elementary school experience. This mindset is that mathematics and paper and pencil are inseparable. Students have never been encouraged to choose from appropriate computational techniques including mental computation, estimation and calculators.

The elusive nature of both estimation and mental computation has resulted in mostly helter-skelter efforts to integrate them into mathematics programs. Evidence of students' performance on estimation indicates that most students do not have high proficiency with it, nor do even good estimators have a strong level of confidence in their ability to estimate. Although both mental computation and estimation have in recent years enjoyed a wealth of support for inclusion in mathematics programs, when all is said and done, much more has been said than done.

What is Mental Computation and Computational Estimation?

Before the role of these topics in a junior high program is discussed it is important that each of them be clearly delineated. There are three distinguishing characteristics of mental computation. It involves the arithmetic of numbers. It produces an exact answer. It is done mentally, without using external devices such as pencil and paper. The distinguishing characteristics of computational estimation are similiar. It is performed mentally, generally without paper and pencil. It is done quickly. It produces answers that are not exact but are adequate for making necessary decisions.

Mental computation is an important component of estimation, since it provides the cornerstone necessary for the various numerical processes used in computational estimation. Typically, the nature of the problem and the numbers involved determine whether the problem is solved with mental computation alone or with a combination of mental computation and estimation. It is possible to be competent at mental computation and yet very poor at estimation. However, the converse is not true; that is, people who are proficient computational estimators have also acquired skill at mental computation (Reys, Bestgen, Rybolt, & Wyatt 1982).

How Should Mental Computation be Organized For Instruction?

Mental computation should be a visible part of an elementary mathematics program. As such, mental computation should be planned for and established early in primary grades and then continued into the junior high school program. Mental computation can be used in the primary grades to build a rationale for algorithm development. It is the bridge between basic facts and algorithms. It is a natural extension of the development of basic facts and actually provides a source of variety in practicing basic facts (e.g., instead of practicing the addition table for 2's why not substitute the 20's?). Within the elementary school, mental computation also serves the dual role of extending place value concepts and serving as an application of otherwise abstract mathematical properties (e.g. 15 x 70 = (10 x 70) + (5 x 70) is the distributive property.) Later it can be used to develop as well as evaluate student knowledge of basic concepts (e.g. What is 50% of 48 or 20% of 100?).

A junior high mathematics program emphasizing mental computation should have the following characteristics:

1. Students should appreciate the importance and value of mental computation. Many students may not be aware of how often they use mental computation, and may not realize that mental computation is used more often than paper-pencil computation in their daily living.

2. Instruction must be built on a framework that provides for systematic development of mental computation throughout primary and middle grades.

3. Instruction must include a variety of different numbers (whole numbers, fractions, decimals and percents) and various operations with them.

4. Specific mental computation procedures must be carefully and meaningfully taught, so that they are understood by all students. Such procedures should be used as a framework to insure that key mental computation strategies are highlighted and then sequenced throughout the mathematics curriculum. Space and time considerations preclude mapping out an entire scope and sequence in this discussion but here are some useful and

powerful mental computation techniques for multiplication of whole numbers.

a. Multiply 2-, 3- and 4-digit numbers ending in 9 by a one-digit number by applying subtractive distribution.

4 keys at \$2.99 4 x \$3 = \$12
but that's 4 cents too much so \$11.96.

b. Multiply by grouping compatible numbers.

$$8 \times 14 \times 125 = 8 \times 125 \times 14$$
$$= 1000 \times 14$$
$$= 14{,}000$$

c. Multiply by building on a known fact.

8×32

$2 \times 32 = 64$
$8 \times 32 = 4 \times 64$
256

d. Multiply by expanding factors

$$6 \times 45 = 6 \times 5 \times 9$$
$$= 30 \times 9 = 270$$

e. Multiply by halving and doubling

$$12 \times 15 = (1/2 \text{ of } 12) \times (2 \times 15)$$
$$6 \times 30 = 180$$

f. Multiply by 5, 50, 500

24×50 $24 \times 100 = 2400$
and 1/2 of 2400 is 1200

g. Multiply by 25 and 250

19×25 think quarters
16 quarters is \$4, so
$19 \times 25 = 475$

h. Multiply by using the difference of two squares.

29×31

$(30 - 1)(30 + 1)$
$900 - 1 = 899$

This is only a snapshot of the many mental computation strategies which should be integrated into the junior high mathematics program. Not included in this list are the more basic techniques of multiplying by multiples and powers of ten. While these are basic techniques, research indicates that we

cannot assume junior high students can apply them. Instruction focusing on techniques for determining place value should be discussed.

The strategies outlined here are straightforward. However, many will be new to students and care must be taken to set up a program which systematically develops, reviews and evaluates students' progress. We suggest that mental computation become a 10-minute warm-up activity on a routine basis (perhaps 1 or 2 days per week).

The techniques outlined point out another important characteristic of mental computation. Not every arithmetic problem will be computable mentally by all students. Mental computation should be encouraged and developed for only a subset of arithmetic problems--only those that can be more easily and efficiently accomplished by mental techniques rather than paper/pencil or mechanical techniques. An important task of the teacher is to help students be aware of these problems as well as of their own ability to mentally carry out techniques such as those described here.

How Should Computational Estimation Be Organized for Instruction?

Estimation, much like problem solving, calls on a variety of skills and is developed and improved over a long period of time. Like problem solving, estimation involves an attitude as well as a set of skills. It permeates many areas of our existing curriculum and to be effectively developed it must be nurtured and encouraged throughout the study of mathematics. Estimation is not a skill which can be isolated within a single unit of instruction. On the contrary, when this is the approach as we have seen in recent years, the effort may even be counterproductive - leaving students with a general dislike and distrust for the very process. To be truly effective, a careful integration of estimation must occur. A comprehensive estimation curriculum must address several areas: development of an awareness for and an appreciation of estimation; development of number

sense; development of number concepts; and development of estimation strategies. All of the components, thoughtfully blended into the problem solving process will not only help students become better estimators but will also strengthen their total understanding and appreciation of mathematics.

A junior high mathematics program emphasizing computational estimation should have the following characteristics:

1. Students should become aware of when estimates are used. Experiences need to be provided to help students differentiate between situations using exact values or estimates. Deciding whether a number is exact or an estimate not only leads to some interesting and exciting discussions in class but contributes to the development of number sense.

2. Attention should be given to helping students decide not only when estimation is appropriate but whether an overestimate or an under-estimate will be most useful. Such decisions can only be made within specific problem contexts, so this will insure that students experience a wide variety of real world applications of estimation.

3. Estimation should be done before written algorithms are encountered. This approach encourages students to "think about the numbers" before performing tedious computation. It won't eliminate all errors from written computation, but it does help students become more aware of what answers are reasonable.

4. Specific computational estimation strategies should be taught. Any program purporting to emphasizing computational estimation should consider including the following key strategies: front-end, flexible rounding, compatible numbers, special numbers and clustering. These strategies are discussed in more detail in Reys and Reys (1983) and Reys (1984).

5. Adjusting is an important part of the estimation process. It complements all of the strategies and provides a means of refining the initial estimate. It is not an algorithmic procedure but is closely akin to number sense. Although students will readily understand the need for adjusting, it may be difficult for them to understand the exact processes they should use to determine an appropriate adjustment. Often the amount of adjustment rests on a sort of basic intuition, i.e., number sense.

6. Instruction must include different numbers (whole numbers, fractions, and decimals) and various operations with them. Although

the initial introduction to a strategy may be with whole numbers, the strategy can be extended and expanded to accommodate estimation with other numbers and operations as appropriate.

7. Attention should be given to helping students develop criteria which help them select strategies appropriate to particular situations. Selecting appropriate estimation strategies is similar to selecting appropriate problem solving strategies. It assumes that a variety of strategies are available and that certain strategies are more appropriate than others in given situations. This type of analysis and decision making requires a high level of thinking.

This highlights a few major components for computational estimation. After these strategies are established, they can be extended and expanded throughout the secondary mathematics program.

Closing

Although some research is reflected in this discussion, much research is needed in both mental computation and estimation in order to direct further curriculum development. Many fundamental questions remain to be answered.

Do contemporary standardized mathematics tests give sufficient emphasis and provide a valid measure of mental computation and estimation?

How should evaluation of mental computation and computational estimation be planned and conducted?

Does an optimal ordering for teaching computational estimation strategies exist?

How much time should be devoted to teaching mental computation? How much time should be devoted to teaching estimation?

How does the development of computational estimation contribute to the development of problem solving?

How can technology be utilized to develop mental computation? How can technology be utilized to develop estimation?

We cannot afford to wait until these questions are answered before integrating these topics into the junior high mathematics curriculum. However any proposed mathematics curriculum should include efforts which might help answer such fundamental questions.

This treatment of both mental computation and estimation has been very brief. In particular it has concentrated on the junior high curriculum and has not attempted to address how mental computation and estimation should be taught. The pedagogy associated with these topics is a very important but different issue. Hopefully this discussion will serve as a springboard for a careful study of how mental computation and estimation should be addressed in the junior high mathematics program.

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PROBLEM SOLVING

Stephen Krulik and Jesse A. Rudnick

Introduction

Problem solving is rapidly becoming an integral part of the school mathematics curriculum. The movement that began with the Position Paper On Basic Mathematics Skills prepared by the National Council of Supervisors of Mathematics in 1978, and the Agenda For Action: Recommendations For School Mathematics issued in 1980 by the National Council of Teachers of Mathematics, has gained virtually unanimous acceptance by mathematics educators throughout the world. At the International Conference of Mathematics Educators held in Adelaide, Australia during the summer of 1984, one of the major strands and several working groups were devoted to problem solving. In the United States programs of local and national professional meetings reveal an inordinately large number of sessions devoted to problem solving. Evidence of this acceptance can also be seen by examining the new generation of basal texts which, in varying degrees, have all included problem solving. In part, this has been brought about by the growing number of state departments of education that have included the topic as fundamental to their approved curriculum. Some have even gone so far as to require all texts seeking state approval to have a major focus in problem solving.

Why is Problem Solving So Important?

Teachers are charged with the responsibility of preparing students to face life in a real world. Teachers of mathematics must decide what skills the students will need in this vital subject. When one considers the fact that students entering school today will live most of their lives in the 21st century, it becomes virtually impossible to predict all of their mathematical needs, particularly with the technological advances

that will impact upon them. It is a certainty that technology will play an ever-increasing role in school as well as in society. This increased role of technology will enhance the position of problem solving in the school curriculum. While the calculator and/or the microcomputer can perform calculations rapidly, it is a human being that must determine what actions the machines must take. Problem solving is the realm of the thinker; solving the problem is the realm of the machine.

One thing, however becomes quite predictable. Students will face problems and will still be required to think logically, creatively, and critically as they are called upon to make decisions. Much of the content currently stressed will probably diminish in importance, but the problem solving skills will surely remain vital.

Even if we were only considering the immediate future, problem solving would still have a high priority. Mathematics skills are useless if they cannot be applied to problem situations. In the final analysis, problem solving is why we teach mathematics. It is the primary BASIC SKILL.

What Is Problem Solving?

Many observers and teachers as well have raised significant questions about the movement. They ask, "What is all the fuss about?", "Haven't we been including this topic in our school problems for years and years?". The direct answer to these questions is NO? What we have been teaching our young students to do for years and years is how to "solve problems", but not how to become "problem solvers". There is a significant difference! Problem solving is a process. It is the means by which an individual uses

previously acquired knowledge, skills and understanding to satisfy the demands of an unfamiliar situation.[1] The process begins with the initial confrontation and ends when the arrived-at answer has been checked to see if the conditions have been satisfied. This entire process is the solution. The development and honing of this process by our students is the goal of instruction in problem solving.

Is this what has been done in school classrooms for years and years? Again, the answer is NO! Emphatically NO! Teachers have not been helping students to develop the process. In fact, in most cases they have merely shown the students how to obtain an answer. They have identified a set of specific problems, each illustrative of a type of "problem". Along with each sample problem, the students were given an algorithm. In other words, students were given problems and shown how to get an answer. Then they were assigned similar "problems" on which to practice the algorithm. This practice was repeated over and over with different types of "problems". The only difference within a given set were the numbers used. The students' main task after learning these algorithms, was to identify the type of problem he or she was faced with, and to recall from memory the appropriate algorithm. It is a gross error to even refer to these "problems" as problems. A problem is an unfamiliar situation. These "typical problems" are only problems when they are initially presented. From then on, they are merely exercises. It can safely be said that teaching students to solve exercises does nothing toward improving their problem solving skills. Of course the word problems that appeared in most

[1] Krulik, Stephen and Rudnick, Jesse A. Problem Solving: A Handbook For Teachers, Allyn and Bacon, Inc., Newton, MA, 1980.

of the older books were not intended to develop problem solving skills. They were included to provide drill and practice for the mathematics skill that had just been presented.

Implications for the Classroom

In order to achieve improved problem solving on the part of children, some current practices in both content and pedagogy will have to be adjusted. Teachers must focus instruction upon the process. This instruction must begin in the primary grades and continue through the entire school, reaching a peak in the junior high school. Student textbooks must have a sequential development of problem solving throughout all grades. Problem solving should not be restricted to a single unit in any text, but must be embedded within the entire text. Problem solving is a process and must be treated as such. As is true with any process, problem solving must be taught, retaught, and used on an on-going basis in order to be mastered. Beginning with the initial encounter and continuing through until the answer has been achieved, examined and extended, each step should be carefully taught, and then used throughout the curriculum. Thus the process permeates the entire curriculum, K-12. The entire process must be emphasized within each grade level. We consider the intermediate grades, six through eight, to be an extremely important time in the development of problem solving abilities in children. Traditionally, this is the transitional period from the computational emphasis in the elementary school to the college preparatory program of the senior high school. It is also the time when children become aware of the world in which they are living. With the rapid advances in technology, many of the topics currently stressed in this time period can be removed from the textbooks of these grades, and can be replaced by formal instruction in problem solving. Naturally, the model problems, strategies, and mathematical processes should be increasingly sophisticated as students move from the primary to the

secondary grades. Instruction in all mathematics must be designed to develop a repertoire of skills and strategies for every student. The instruction should also be designed to increase the level of proficiency required in applying these skills and strategies. Instruction should be carefully structured in order that this process will become a natural part of a student's attack plan in solving any problem.

The emphasis in teaching problem solving must be on the process. The use of calculators and computers should be incorporated into the curriculum in order to facilitate obtaining answers. Using these electronic tools will create additional time in the curriculum, as the amount of time allotted for drill exercises can be decreased. In addition, these tools allow for more realistic data as part of the problems being presented. By including realistic data, the motivational level for the students is increased since it makes the classroom mathematics more "real world". Situations that are related to the students' everyday life should be used as part of the problem solving instruction. Some problems should be structured to require the students' mathematics imagination and creativity to be stretched to the limit in order to find the answer. Problems that cause the students to think about mathematics and how it can be applied are vital. This thinking dimension of mathematics is where the true impact must be made, and the curriculum and the delivery of instruction must include situations that cause it to happen.

Research has found that students do not naturally develop problem solving abilities, and that, with proper instruction, a student can learn to think more logically and creatively. The curriculum and the delivery of instruction must reflect a high level commitment to the use of this process skill called problem solving. It has to become natural for each student. It therefore must be a priority instructional goal for all mathematics educators as they plan

for the best possible mathematics educational interests of the students of tomorrow.

SPATIAL VISUALIZATION IN THE MATHEMATICS CURRICULUM

Grayson H. Wheatley

While most school curricula stress rules, procedures, and sequential reasoning, there is much more to learning mathematics. Many important ideas require the construction of images and the manipulation of these self-generated mental pictures. Yet, where in the mathematics curriculum do we encourage the development of imagery? We require students to memorize facts and formulae but where do we help them make mental pictures? We ask children to analyze but how often do we have them visualize? It seems as though we are failing to develop a powerful and essential mental faculty and inadvertently, failing to recognize the strengths of many of our image makers.

While it may be true that the development of language and analytic thought has been the springboard for major advances in civilization, we must not forget the great power of image-mediated thought. The power of imagery is seen in the lives of many creative persons. Eminent scientists have frequently reported how discoveries came in flashes of insight which are heavily image laden. There is insufficient space in this paper to describe the many examples of this phenomena, however, consider Einstein and the theory of relativity, Kekule and the benzene ring, Watson and Crick and DNA. In each of these examples imagery, not logical thought was the key to the discovery. Learning mathematics (not rules) is also a process of discovery; students must construct, frequently using imagery, concepts and relationships.

The case for spatial visualization as a key component in mathematics performance is quite strong (Guay and McDaniel, 1977; Frank, 1979; Sharo,

1962; Sherman, 1979; Smith, 1964; Turner, 1982). Of special interest is Turner's study in which he found that persons performing well in calculus generally exhibited high spatial ability and preferred to think in images. We can greatly enhance students mathematics performance if opportunities to develop spatial visualization are integrated into the school mathematics curriculum.

Spatial Visualization and Problem Solving

At present there is enormous interest in problem solving. Numerous problem solving materials have been developed and textbook publishers are even making available supplementary booklets on problem solving. The role of imagery in problem solving is quite strong (Wheatley, 1977). Consider for a moment the way spatial visualization comes into play in problem solving. As a problem is read, if it is presented in written form, the solver creates a mental picture of the interrelationships. Yackel (1984) and Wheatley and Wheatley (1982) have shown that building a mental representation of a problem is a major factor in successful problem solving. This is surely not surprising to mathematics educators. If students are weak in forming mental images or do not believe that is the appropriate attack, they are likely to be limited in their problem solving efforts.

A heuristic on everyone's strategy list is draw a diagram. Teachers have for years implored their students to draw a picture. Recently, the importance of teaching this heuristic has been recognized and impressive results have been obtained. Why do students initially have such difficulty drawing a diagram of information in a problem? Poage and Poage (1977) report evidence, of what most of us know, that students typically do poorly on problems involving diagrams. For example, in the Wheatley and Wheatley study 102 sixth grade students were interviewed as they attempted to solve

the following problem.

> Paul has a swimming pool in the shape of a rectangle. It is 31 feet long and 23 feet wide. There is a walkway 3 feet wide around the pool. What is the length of a fence around the walkway?

Less than 12 percent of typical sixth grade students drew a diagram in attempting to solve the problem. Why? When, as adults we 'naturally' draw a diagram for this situation it is important to recognize that the diagram does not originate on the paper; it must first be created in the mind. Before one can draw a picture, that picture must first be formed mentally. We can only express what we know. But for far too long 'knowing' has not included imagery. How often have we heard it said, "You don't understand it until you can explain it." In this context 'explain' means put in words but many ideas are not easily or even best expressed in words. The old adage, "A picture is worth a thousand words" rings true.

Components of a Spatial Visualization Strand

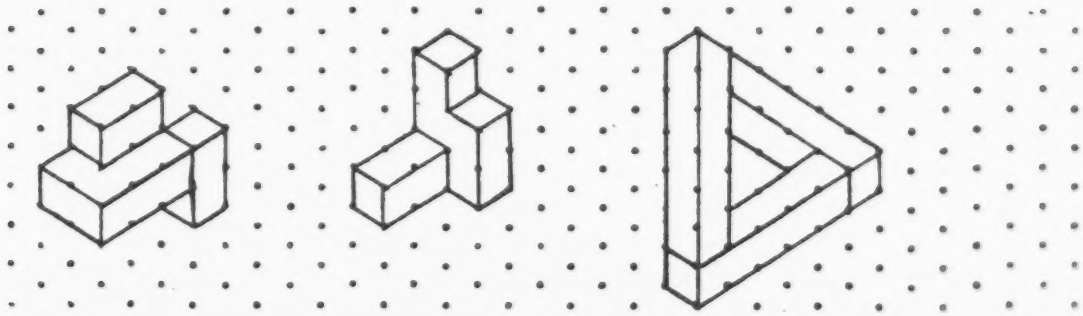
A pre-algebra course should provide a wide variety of opportunities for spatial representation of ideas. Activities for developing spatial imagery include, but are not limited to these.

1. **Constructing three dimensional shapes from cardboard.**

   The five Platonic solids provide a good start.

2. **Drawing three dimensional shapes.**

   Isometric dot paper is excellent for this purpose as described in Lappan, Phillips, and Winter, 1985 and shown below.



3. Drawing objects from memory. The following activity is designed to encourage image building.

Picture Your School

Goal: Develop the ability to visualize what cannot be seen.

Rationale: One aspect of functioning in our world is the ability to visualize objects from a different perspective. In this activity, the students are asked to "look down on" their school. While students frequently go in and out of the building they most likely have not thought what it looks like from the air. The act of visualizing and then refining their image encourages the development of spatial ability. Further, it may sensitize them to viewing objects from different perspectives.

Activity: Draw a picture of your school.

a) Without any introduction, ask pupils to use a full page and draw a sketch of their school viewed from above. Ask them to imagine the school layout. Allow at most 10 minutes for this activity. Use the remaining part of the period to work on a different activity.

b) Collect papers and discuss the purpose of the activity and the importance of spatial visualization. Indicate that you will ask them to draw another sketch tomorrow and thus they may wish to think about the shape of the school as they walk around in it.

c) Next day ask them to again draw a of sketch their school.

Collect papers and discuss. Assignment: Walk entirely around the school and make mental notes of its shape.

d) Next day ask them to again draw a sketch of the school. This time have them keep their sketch and work at refining it. Have students in teams pace off various dimensions of the school so that proportional drawings can be made. Encourage students to put in as much detail as they

wish but as a minimum draw the outline as accurately as possible.

e) After sufficient time, distribute students' first sketches for comparison with their finished product. Collect final drawings, evaluate, and post the best ones in the classroom. Variation: Have students explore a two or three story building and draw floor plans of each level. Then ask them to visualize the building and identify which rooms are above a specified first floor room. While in a room, have students identify the direction of key community landmarks such as the county courthouse.

4. Tangrams and Supertangrams

The new Supertangrams commercially marketed are a more complex set of tangram-like tasks which will provide a special challenge for junior high students. The Supertangrams are the 14 pieces that can be made with four isosceles right triangles. Can you find the 14 shapes?

5. Soma Cube

The Soma Cube is a three-dimensional extension of the planar activities with tangrams.

6. Tessellations

Among the variety of spatial activities, I favor tessellations because of the high demand of visual anticipation necessary to imagine rotations. Presently, the topics listed above might be used occasionally by a few teachers. I recommended that at least two full weeks be devoted to this strand with careful attention to building imagery in each unit of the course. This would be a departure from current practice but a much needed change.

Summary

School mathematics curricula is a slowly evolving set of ideas. Conventional courses are strongly influenced by behaviorism, with mastery

of skills by practice the major pedagogical method. This line of reasoning has led to linear, symbolic, and analytic thought as the only mode of mental processing. Some students view the world in terms of images and have great difficulty with the computationally oriented mathematics courses. Other students have latent spatial ability but do not have the opportunity to develop and use it. For many reasons it is time to restore the balance by including spatial visualization in the curriculum. It will pay dividends many fold for the time invested.

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UTILIZING APPROPRIATE TECHNOLOGY IN THE LEARNING AND TEACHING OF MATHEMATICS

Bob Kansky

While the instructional implications of all forms of classroom technology (e.g., slides, films, transparencies, demonstration devices, laboratory materials, television, calculators, computers) demand continuing attention, the curricular prescriptions to be presented in this paper will focus upon the uses of microcomputer systems as tools in the learning and teaching of mathematics. Those prescriptions assume that both calculators and computers are, or soon will be, readily available to all junior high school students, and that calculators and computers should be employed as standard mathematical tools during classwork, homework, and evaluation activities.

Most junior high school students do have access to calculators. As early as 1978, NAEP data confirmed that 80 percent of thirteen-year-olds either owned a calculator or had one available for use. (Carpenter, 1981) It should now be a small matter for schools to supply calculators (in the same manner as textbooks) for those who cannot afford to purchase them. Certainly, further hesitation owing to concern for possible negative outcomes resulting from the use of calculators is not justified. Most studies show no significant differences in the achievement measures of calculator and noncalculator groups or show differences in favor of the calculator group. (Suydam, 1980, 1981) It has been suggested that the lack of a body of evidence clearly favoring the use of calculators within secondary mathematics classrooms is probably due to the trivial tasks to which calculators are applied in many studies (e.g., checking answers), the failure to develop instructional materials which build upon the strengths of calculators, and the limited attention given to integrating calculators

into the curriculum rather than using them in supplementary activities. (Roberts, 1980) Relevant research activity, declining in volume after a period of intense interest prior to 1980, has provided evidence that calculators enhance the learning of both problem-solving and basic paper-pencil skills. (Driscoll, 1982; Hemree & Dessart, 1986).

Until recently, student access to computer technology has been defined in terms of access to microcomputers. The purpose of such access typically has been drill activity or the study of the computer itself (as part of units on programming or computer literacy) (Becker, 1985). The introduction of hand-held pocket computers with graphics capabilities and the companion instructional intent of using the computer as a tool in problem solving demand redefinition of "access" and give credence to the assumption of ready availability in the near future. Unfortunately, given this new technology and new focus for the use of the computer in the mathematics classroom, a research base for the suggestions to be made here simply does not exist. Generalizations suggested by meta-analyses of research on instructional computing are based upon studies most of which (a) predate microcomputers, (b) were executed on postsecondary populations, and (c) investigated tutorial, drill, or practice applications (Krulik, Bangert, & Williams, 1983; Krulik, Krulik, and Cohen, 1980). For the moment, then, curriculum-wide integration of computers as problem-solving tools must be guided by best guesses and creativity.

Of the several ways in which technology may impact upon the mathematics curriculum, only two will be discussed here. The first is a consequence of the belief that it is mathematics itself (rather than the textbooks or curriculum guides) that suggests the form of technology best suited to the investigation of a given problem. Since the problem solver is given the charge of recognizing that "suggestion," criteria will be

offered to help in matching mathematical tasks with appropriate technological tools. The second issue, the use of the microcomputer to aid mathematical problem solving, will be addressed through the discussion of categories of software tools.

Appropriate Technology as a Function of the Mathematical Task

The computational component of the pre-algebra curriculum calls for the use of three forms of technology: calculator, computer, paper-pencil. Each form is the best choice for certain applications; no form is superior for all applications. To make "appropriate use of technology", the student must know more than <u>how</u> to use any item of technology--he or she must be able to decide <u>which</u> item is best to use in any given situation.

<u>Computer Tasks</u>

Four criteria might be applied to identify computational situations calling for a computer.

*The student knows how to interpret the results obtained; and

*a large number of repetitive calculations or comparisons is required to test some mathematical idea or pattern; or

*a large data set will be used repeatedly over a period of time; or

*a detailed textual or graphical record of intermediate or final results is needed.

The first of these criteria does not ignore the broader principle that <u>any</u> numerical result, regardless of how it is obtained, should be interpreted in terms of the original problem situation. Rather, it is meant to direct attention to the student error of reporting results with false precision, a problem that has been compounded by the ease with which a calculator (or computer) can generate figures to several decimal places. For example, a student who has measured the legs of a right triangle and reported them as 31 m and 44 m should realize that it makes no sense to report the

calculated length of the hypotenuse of that triangle as 53.823787 m . The use of calculators, coupled with greater curricular attention to problems which involve real data (i.e., measurements), will require increased attention to the reasonableness of results in terms of rules that govern calculations with approximate numbers. Instruction which promotes understanding of the approximate nature of measures (Kerr & Lester, 1976) must be augmented by the development of guidelines for "rounding" the results of calculations performed upon approximate numbers.

With the exception of the first criterion, the criteria are to be applied disjunctively. Thus, once it is assured that the student is able to interpret the results of the calculations, any one of the other conditions would be sufficient to justify the application of a computer. The repeated use of an algorithm, repeated use of a data set, or generation of long/complex output are tasks requiring prodigious feats of memory, speed in applying a well-defined procedure, and unflagging attention to tedious details--requirements perfectly matched by the fundamental talents of a computer.

Example: While the initial investigation of a mathematical idea might proceed nicely with the aid of just pencil-paper or a calculator, it sometimes becomes evident that the resolution of a question depends upon the repeated application of a multi-step procedure. Consider this problem.

The marching band at Very High School decides to sell earmuffs ($3 profit each) to earn the $2400 needed to pay its way to the state tournament. Two local service clubs agree to help. One club offers to contribute 50 percent of whatever the band takes in from earmuff sales or from other contributors; the second club offers a 75 percent contribution under the same terms. Band members sell 204 earmuffs. Is the help promised by the two service clubs enough to allow them to make the trip? If not, what minimal number of earmuffs must be sold to make the trip possible?

Although elementary algebra could be used to organize a solution to this problem, a guess-and-compute strategy will do the job. In either case, the

calculations required during repeated executions of the strategy are burdensome even when done on a calculator. A computerized form of the algorithm is the tool needed.

Putting the Microcomputer to Work in the Mathematics Classroom: A Problem Solver's View

Thus far we have focused attention on criteria for matching items of computer technology with mathematical tasks. Once the computer is identified as the appropriate tool, the next task is to match the problem solver (student or teacher) with appropriate software tools. These software tools of the mathematical problem solver can be sorted into at least three categories: student-generated software tools, teacher-generated software tools, and commercially-prepared applications software tools.

Student-generated software tools are short programs, designed and encoded by the student, that carry the burden of performing repetitious numerical or graphical manipulations incidental to an ongoing mathematical investigation. The computing language employed should match the task; hence, BASIC could be used to resolve a host of computational tasks, but Logo might be a better choice for exploring geometric conjectures. Whatever language is chosen, the subsequent programming effort should be governed by two adjectives: short and simple. That is, any programming activity should be a brief and purposeful digression within some larger problem-solving context. The program (tool) to be created is but a means to a mathematical end. It follows that the student's programming skills need not be extensive but, if they are to be employed in a variety of problem situations, probably need to be developed using more than one computing language.

Teacher-generated software tools serve to bridge the gap between the simple products designed by students and sophisticated applications packages developed by professional programmers. Like student-generated software

tools, these programs are specific to the investigation of particular mathematical ideas. They differ from student-generated tools in that their construction would require the development or exercise of student programming expertise judged to be disproportionate to the mathematical problem-solving tasks to be served by them. Thus, although the general structure of any one of these problem-solving tools might be understood by the student using it, instructing the student in the language (or level of language) in which it can be efficiently encoded would divert instructional focus from mathematics to programming. And while it is obvious that a teacher's programming skills place an upper bound on the number and type of custom-made tools that can generated, the guideline of proportional effort should be applied to the teacher's programming activity as well as the student's. If mathematics instruction is to be well served by computers, the electronic tools that a teacher should develop will be carefully selected from among those that the teacher could develop. When the development of the tool calls for a large investment of teacher time, the appropriate decision probably involves the purchase of an application software tool.

The term 'applications software tools' is used to denote a set of complex, flexible commercial programs which manipulate, transform or display data or mathematical expressions. Examples of application tools include databases, word processors, graph-generation packages, spreadsheets, statistical packages, and packages which assist in the construction, transformation, or manipulation of geometric shapes. Each such tool is of practical value, however, only insofar as the range of problems to which it can be applied is sufficiently large to justify the amount of problem-solver (teacher or student) time that must be invested in learning how to use it. This criterion of practicality also is a function of the level within the teaching-learning process at which the application is made: curriculum

development, classroom instruction, or student use.

Curriculum Development Tools

A textbook-based curriculum is harnessed to an adoption schedule of three to five years. Except for minor corrections introduced during reprintings, textbooks themselves are locked into a five-year rennovation schedule by the extreme expense of substantive revision. Teachers seeking to update or enliven such a curriculum currently are faced with the task of locating and/or modifying supplementary materials which match the interests and abilities of their students. Maintaining such a resource file and using it effectively can demand considerable time and space. It can, but it need not.

Consider the task of infusing problem-solving activities into all strands of a junior high school mathematics curriculum. The resource needed is more than a large collection of appropriate problems. It is a collection in which each problem is categorized by such use variables as mathematical strand, problem-solving strategy, instructional intent (e.g., to develop a new topic within one of the strands or to take a given concept to a higher level), instructional extent (isolated task, series of tasks, project), student grouping (individual, pair, small group, whole class), and technological tools needed. Electronic curriculum development can utilize computer integration of a word processor, graphics package, database, and communications package to put that collection of problems (and related teacher-support materials) at the fingertips of the classroom teacher. The sorting power of the database can be used to locate problems needed to fit a particular application. The word processor and graphics utility offer the teacher the opportunity to "personalize" the wording of problem situations (say, by changing the names of persons or places involved the problem

statement), to alter out-of-date data, or to modify the problem to suit a different audience. Modification of the primary set of problems can be carried out instantaneously; regular revisions of the resource can be delivered quickly with the aid of the communication system; bulletin boards and mailboxes on the same system might be used by teachers and students wishing to share new problems and/or problem-solving ideas. [In fact, a consortium of universities has proposed to develop this electronic curriculum resource.]

Classroom Instruction Tools

Mathematics teachers interested in solving instructional problems will find it profitable to learn to operate several applications tools. When used in conjunction with a computer-projection system, these tools can enhance whole-class instruction in what might be termed a demonstration mode of computer use (Dennis & Kansky, 1984). For instance, graph-generation packages can be used to match classes of relations with their characteristic graphs by quickly and accurately plotting representive families of those relations on each of several kinds of (electronic, on-screen) graph paper. Conversely, the plotting of experimentally-generated data points on several kinds of graph paper can lead to the identification of the class of relations which best fits the data. Use of a spreadsheet in the demonstration mode can support the classroom game of "what-if" as used to explore, say, properties of specific sequences. Statistics packages being designed for the middle school and high school can be used to conduct statistical inquiries by generating several alternative graphical summaries or numerical analyses of a given data set. The demonstration mode also can be applied to programs which aid in the construction, transformation, or manipulation of geometric shapes. Such demonstrations can be used to provide students with opportunities to conjecture and test geometric properties. Individually, demonstrations

familiarize students with the operation and problem-solving possibilities of a particular application tool; collectively, demonstrations provide students with experiences in planning and executing useful problem-solving strategies.

Tools for Student Use

Teacher uses of applications tools should lead to comparable independent uses of those tools by students engaged in problem-solving activities. Independent use does not require that the teacher instruct each student in the use of each electronic tool; neither does it require that each student develop high facility in the use of each tool. The desired independent use can be realized through the use of "teams" of problem solvers within which individuals take responsibility for learning to use particular tools and subsequently assisting other members of the team to apply those tools to investigate mathematical ideas. In fact, research on cooperative learning supports the assertion that the use of academically and socially heterogenous groups of five or fewer students will produce higher levels of achievement on complex tasks such as problem solving than will competitive or individualistic learning environments. Peer teaching within the cooperative learning groups not only promotes positive interdependence among the members of the group but also affirms the individual accountability of each member of the group (Johnson, Johnson, & Maruyama, 1983; Johnson, Maruyama, Johnson, Nelson, & Skon, 1981; Slavin, 1980).

In addition to making use of the application tools previously described, students should become familiar with computerized "tool kits" prepared by mathematics educators to enhance study within standard secondary school mathematics courses. Kimberling (1984a, 1984b), for instance, has prepared collections of programs which afford students (and teachers) the opportunity to apply the computer in the investigation of uncommon questions about such

common topics as primes, factoring, fractions, random numbers, and graphs of selected relations. Unlike the generic application tools described earlier, these are course-specific tools. Essentially collections of teacher-generated software tools produced by a master teacher, these tool kits support a variety of mathematically relevant, programming-free problem-solving experiences. Since the programs are open to editing, however, a student or teacher with modest programming skills could modify or build upon any program.

New Learning Roles for Students and Teachers

The goal of having students make "independent use" of computers to solve mathematical problems does not imply a learning environment in which the teacher is responsible for developing the technological expertise of each student and in which each student works in isolation to apply that expertise to the solution of mathematical problems. Rather, the application of the computer as a tool in the study of mathematics suggests the need to develop new student-teacher and student-student roles.

> *Teachers must serve as learning models in attacking mathematical questions. They must be comfortable assisting students in learning situations in which the topic being studied or the resource employed is unfamiliar. Teachers must be prepared to say "I don't know yet," while they join students in exploration.
>
> *Teachers must become facilitators of cooperative learning as students engage in realistic learning projects that computers make attractive. Teachers and students must emulate real problem solvers as they deal with problems of complexity, accuracy, and precision that occur in the collection and analysis of real data.
>
> *For students, the greatest promise and challenge of technology in mathematics learning is the move toward more self-directed and self-monitored learning. Students must become adept at using computer-based instructional materials, mathematical tool software, and personal programming skills to develop and practice the use of mathematical concepts, principles, and problem-solving processes. (Corbitt, 1985, p. 247)

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IMPROVING DEVELOPMENT LESSONS IN MATHEMATICS BY CHANGING TEACHERS' VIEWS OF MATHEMATICS AND THEIR INSTRUCTIONAL PRACTICES

Thomas L. Good

As an educational psychologist, I have for some time been interested in the effects of classroom instruction (e.g., teacher-student interactions, assigned tasks) on students' thinking and classroom performance. In particular, much of my work has examined how teacher expectations for student performance (individual students, groups of students, or even whole classes) affect student performance. My research has focused on the negative aspects of expectations and on how teachers communicate low performance expectations (Brophy & Good, 1974; Cooper & Good, 1983; Good and Weinstein, 1986).

I have become convinced that because of inappropriately low performance expectations, some teachers communicate with certain students so as to reduce or sustain their performance. Table 1 illustrates some of the ways in which this communication process has been found to work. Table 2 shows that although teachers' expectations may reflect different teaching styles (Model I represents an overly critical teacher; Model II depicts an undemanding teacher), both models depict low performance expectations associated with tasks and communication that lead to less student thought and effort.

In this paper I discuss a different type of expectation and explore its implications for students' classroom performance. I shall argue that many teachers' conceptualization of mathematics emphasizes computation (the ability to obtain correct answers quickly), a view that is supported if not enhanced by textbooks. As one way of broadening their definition of mathematics, I will argue that teachers need to pay more attention to

the development portion of instruction--to helping students understand mathematics, not merely perform it.

Belief About Mathematics

For a long time, mathematicians used mathematics as a way of representing true statements about nature. Mathematics was a means of knowing, a way of achieving certainty that relied on deductive proof from self-evident principles called axioms. Deductive reasoning by its very nature guarantees the truth of what is deduced--if the axioms are true. As Kline (1980) notes, by utilizing this seemingly clear, infallible logic, mathematicians produced apparently irrefutable conclusions. Although mathematicians no longer believe in this feature of mathematics, it is still cited today, and many people continue to believe that mathematics produces irrefutable conclusions. When writers want to provide an example of exactness and certainty of reasoning, they often refer to mathematics.

In his fascinating historical account of beliefs about mathematics, Kline (1980) points out that the current predicament of mathematics is related to the fact that there are many conceptions of mathematics, and that for numerous reasons, each conception fails to satisfy members of opposing schools. He documents how the conception of mathematics as a universally accepted and infallible body of reasoning has developed into the field of uncertainty and differential perspectives. Indeed, as Kline notes, uncertainty and doubt concerning the future of mathematics and its application has replaced the certainties and complacencies of the past. Mathematicians have to accept the view that mathematics is an imperfect discipline. Although one can speak of the uncertainty of mathematics, it is clear that this subject continues to be an important force in society.

The reasoning and organizing properties associated with mathematical

thinking are helpful in making one more aware of alternative conceptualizations of problems and alternative pathways to problem solutions. Most mathematics involves providing estimates and approaches to problems rather than simple quantitative statements. However, most citizens and too many teachers view mathematics merely as producing correct answers. This is an unfortunate view that helps to account for the fact that mathematics teaching in elementary schools basically involves arithmetic computation and drill, with relatively little attention to problem solving or estimation.

Students' Mathematics Achievement

An examination of student performance on mathematics achievement tests supports this contention. The most recent test results (see National Assessment of Educational Progress, 1983) show that students demonstrate a high level of mastery of computational skills, especially those involving whole numbers. In all other areas, however, many students are not learning some of the most basic mathematical concepts and skills. Such findings have led researchers to conclude that if students' performance on the National Assessment is a measure of instructional emphasis, then most mathematics programs focus on the development of routine computational skills (Carpenter, Matthews, Lindquist, & Silver, 1984).

I believe that if student performance is to be enhanced, changes in thinking about mathematics must be incorporated into the mathematics curriculum. More general, societal changes that transform how people use mathematics are also needed. The mathematics used in the world today is not the same as that used or needed a century ago. However, the mathematics taught in schools today is generally the same mathematics that was taught decades ago. Although today's students must be able to compute efficiently,

educators need to examine carefully the level of computational proficiency necessary and to determine which skills should be emphasized. For example, inexpensive computers can do long, tedious calculations more quickly and generally more accurately than can an individual using even highly developed computational skills. Thus, it seems unnecessary to spend considerable time helping students learn to calculate long columns of numbers. Nevertheless, observation in mathematics classrooms indicates that students spend a great deal of time working on computational problems.

Because computers can perform complex computations, the relative importance of estimation skills has increased (e.g., to assess the plausibility of answers obtained with a calculator). Unfortunately, estimation was one of the areas in which students performed most poorly on the NAEP assessment.

The National Assessment (1983) contained exercises in a number of basic skills including algebra, geometry, measurement, probability, and statistics. In general, performance was high on exercises involving knowledge and skills, such as tasks often emphasized in textbooks and instructional programs (e.g., recognizing common geometric shapes, using a ruler to make linear measurements, etc.). However, the performance of students at all ages was low on exercises assessing more complex concepts and skills that require understanding of underlying concepts (Carpenter, Matthews, Lindquist, & Silver, 1984). For example, fewer than 10 percent of the 13-year-olds could use the information that the sum of the measures of the angles of a triangle is 180 to find the size of one angle when the sizes of two angles were provided. Such findings indicate that more emphasis needs to be placed on understanding mathematical concepts. Particularly disturbing was the finding that students often could do a simple computation but were unable to estimate the answer.

There is evidence of some improvement in students' performance over the past decade (comparing the 1973 and 1982 National Assessment results). However, the largest gains are on topics that students learn as isolated skills. The evidence suggests that American schools have taught only a small percentage of students how to analyze mathematical problems or apply mathematics to non-routine situations. Students cannot apply basic mathematics concepts they have memorized and perform especially poorly in the areas of estimation and problem solving.

Studies of Mathematics Teaching

Lack of Development

Several recent studies show that elementary-school mathematics instruction is textbook-driven and emphasizes drill. For example, in an information observational study of 37 fourth-, fifth-, and sixth-grade classrooms in nine schools, Doug Grouws concluded that too little time is spent on development, although several studies recommend (on the basis of solid empirical evidence) that 50 percent of class time be spent on development (see Good, Grouws, & Ebmeier, 1983 for a review of this literature). Grouws and Good found that when they defined and measured development as the time spent introducing and explaining mathematics concepts, only 14 percent of the class period was utilized for this purpose. Thus, students have little chance to understand mathematics. Although students are more likely to understand and remember important points if they know what to look for, teachers infrequently attempted to communicate lesson objectives to students.

In addition, there were problems with the quality of what development was observed. Most development lessons focused on memorization rather than on understanding and visualizing concepts or making generalizations. References to everyday situations that students could understand were

infrequent and tended to be relatively trivial and uninteresting. Furthermore, teachers gave too much attention to procedural detail and too little to understanding mathematical concepts.

There was little instruction in problem solving and estimation. Students were not encouraged to question, experiment, explore, or suggest explanations. Additional instruction in the areas of reasonableness of answers, checking results, estimation of answers, mental computation, and making up and solving verbal problems seems warranted, since these important skills were rarely taught. It seems quite clear that to the extent that these classrooms represent normative practice, greater attention to problem solving and the application of mathematics are important instructional goals.

Emphasis on Procedural Detail

These observations about mathematics classrooms have been replicated in other studies. For example, in a case study of a well-managed and well-taught tenth-grade high-achieving geometry class, Schoenfeld (in press) found that students were able to do well on standardized achievement tests but learned virtually nothing about the mathematical ideas underlying the subject matter in the course. Students developed perspectives regarding the nature of mathematics that were not only inaccurate but were likely to impede the students' future acquisition and use of mathematical content.

I will comment here on only two of several problems in this class that Schoenfeld discusses. First, students in the class came to perceive that the form of a mathematical answer was most important. For example, the two-column format in presenting a proof is a matter of convenience; however, in many classes it has become the expected form. Indeed, Schoenfeld notes that in all the classes in which he observed--including the target class--as well as on the New York State Regents Exam, respondents' arguments must be

presented in two-column form. Substantial partial credit is deducted for deviations from this form on exam answers.

It is interesting but regrettable that a great deal of class time was spent considering what forms are acceptable for presenting proofs (do whole statements about corresponding parts of congruent triangles have to be written out, etc.). In some class sessions more time was utilized to discuss format than to discuss <u>results</u> being proved. Schoenfeld analyzed videotapes and found that students would frequently solve a proof problem quickly but would then spend three to five times as long making sure that the argument was written in final form!

Another problem that Schoenfeld reports is that students believe that all problems can be solved in just a few minutes. The teacher usually had students present solutions to as many problems as possible at the board. Students were often expected to work through 25 or more problems in a 54-minute class period. Furthermore, a uniform exam in Math 10 Geometry taught at the school contained 25 problems; students had an average of slightly more than two minutes to work on each problem. Schoenfeld notes that the teacher's advice to the students summed up this perspective: "You'll have to know all your constructions cold so you don't spend a lot of time thinking about them."

Schoenfeld's analysis is similar to my own in one respect--the identification of considerable <u>curriculum clutter</u>. I have long been impressed by the fact that the procedures of instruction and the substance of instruction are often poorly distinguished and that procedural details dominate instruction. For example, in one class that I recently observed, the teacher constantly corrected students, not because they had failed to work problems but because the form they used was wrong (in this case their flow chart/diagram of the problem-solving process was not in proper format or

they had used incorrect symbols, etc.). During much of the class period the teacher focused on this rather trivial issue. Although, the purpose of diagramming problems was to help students understand and solve problems, students in the class were confused. They had to work hard to learn how to do the diagramming, even though they knew how to compute solutions to the problems.

Curriculum clutter is rampant in elementary school mathematics instruction and unless teachers have a clear conceptions of what they teach and why, students will continue to receive instruction that fails to stress key concepts or that appropriately aligns curriculum goals and instructional processes (Kilpatrick, 1985). Textbooks reinforce the fragmentation of procedure and substance by failing to include information relevant to understanding concepts and by presenting numerous brief, insignificant reviews.

Infrequent Instruction in Problem-Solving

Burn and Lash (1984) conclude from a review of the literature that problem-solving instruction rarely occurs. In their own research, Burns and Lash (1984) requested that nine seventh-grade mathematics teachers teach problem solving for one school week. Teachers could use a variety of activities and problems. The only requirement was that they focus on four skills that were presented in the unit. Burns and Lash found that the teachers had limited knowledge they could use to design the unit on problem solving. They considered teaching techniques very little, if at all, during planning of the problem-solving unit; rather, teachers focused on the problem-solving skills or on mathematical content. Planning focus was a major factor in determining whether the skills were specifically taught or whether students simply worked word problems. Finally, instruction during

the problem-solving unit was almost identical to regular instruction except that word problems replaced computation problems. The textbook was of foremost importance in determining the mathematical content and the approach to teaching that was followed.

Burns and Lash (1984) conclude that problem-solving instruction was similar to regular instruction because teachers have a characteristic view of mathematics instruction as drill and review oriented. They contend that any suggestions to teachers for new curriculum or instruction are affected by teachers' views of mathematics instruction. That is, if one attempts to improve mathematics instruction by asking teachers to spend more time on problem solving, teachers will merely assign students more word problems. According to Burns and Lash, if teachers are to change their mathematics instruction, they must receive training on how to integrate basic skills instruction in problem-solving instruction. Furthermore, if teachers are to alter their approach to teaching problem solving, they must be given instruction in appropriate problem-solving techniques.

Another recent study of mathematics teaching raises important questions. Bourke (1984) coordinated the International Educational Achievement Classroom process study in Australia. He examined the relationship between student and teacher backgrounds, classroom contexts, teaching practices, and student cognitive and affective outcomes. Instructional processes in 75 fifth-grade math classes were carefully analyzed. Bourke found that higher-order questions and applications were extremely rare in these classrooms.

Conclusion

Thus, I argue that teachers' and textbooks' views of mathematics as characterized by certainty (help students to do problems quickly and accurately) correlate with and sustain (if not cause) the poor student performance on recent national assessments. Fortunately, there is growing

pressure to modify instruction and to encourage students to understand mathematical thinking (e.g., Silver, 1985). As Grouws (1985) argues, however, it is difficult for students to use mathematical ideas in problem-solving activities if they do not understand those ideas. Grouws contends that the ability to solve mathematical problems is related to problem content. However, mastery of the prerequisite skills to solve a problem does not necessarily mean that a student can solve the problem. How new knowledge is stored and related to previously acquired knowledge may determine how students use new knowledge to solve a problem. This suggests a relationship between how mathematical ideas are developed and presented originally to students and how they use them during subsequent problem-solving activities. I believe that at least one step that needs to be taken if mathematics instruction is to be improved is for teachers to focus on the development of mathematical ideas (Good, Grouws, & Ebmeier, 1983).

Past Improvement Efforts

In our previous work in elementary schools, we have found that many teachers did not regularly use an extended development component in their mathematics lessons. This is unfortunate because it is in the development portion of the lesson where students come to understand mathematical ideas and their value in real-world decision making. Hence, in our present work we are placing special attention on the role of the development lesson in a successful mathematics program.

As we have defined elsewhere (Good, Grouws, & Ebmeier, 1983), the development portion of a mathematics lesson is that part of a lesson devoted to increasing comprehension of skills, concepts, and other facets of the mathematics curriculum. For example, in the area of skill development, instruction focusing on <u>why</u> an algorithm works, <u>how</u> certain skills are

interrelated, what properties are characteristic of a given skill, and means of estimating correct answers should be considered part of development work. In the area of concept development, developmental activities would include initial instruction designed to help students distinguish the given concepts from other concepts. Also included would be association of a label with a given concept. Attempts to extend ideas and to facilitate transfer of ideas, particularly to application situations, are part of developmental work. Thus, particularly to application situations, are part of developmental work. Thus, the development in mathematics instruction is a process whereby teachers facilitate the meaningful acquisition of an idea by learners. Meaningful acquisition means that an idea is related in a logical manner to a learner's previously acquired ideas in ways that are independent of a particular wording or special symbol system.

Criteria for Development Lesson

We identified five components that we believe to be important parts of development. It should be noted that not all components of development will necessarily occur in any particular lesson ... likewise, the existence of one component does not guarantee that development is taking place. However, the following criteria provide a useful way of thinking about the role of development in students' mathematics learning.

Attending to prerequisites. In mathematics an understanding of most new concepts and processes is based upon previously learned ideas and skills, although the prerequisite ideas/skills vary depending on what is to be learned. For example, it would be unlikely for a pupil to acquire meaningfully the notion of a prime number without some knowledge of factors. Similarly, proficiency in using the traditional long division algorithm is not possible without subtraction skills.

Attending to relationships. Mathematics is composed of a large body of

logically consistent, closely related ideas. To teach it as a collection of isolated facts to be memorized (as is often done) is a disservice to the learner and to the discipline. Relationships among mathematical ideas are numerous and the development portion of the lesson provides the teacher with an opportunity to emphasize the meaning and interpretation of mathematical ideas. Some mathematical relationships are simple, such as the doing-undoing relationship between addition and subtraction, or the relationship between angle measure and side length of a triangle. Other relationships are more complex: Is the product of two numbers greater than their sum? Is there also a "next" fraction as there is with whole numbers? As part of the development the teacher must bring into focus the interrelatedness of mathematical ideas.

Attending to representation. Mathematical formulas, theorems, processes, and indeed most of mathematics, are a step removed from the physical world. This level of generality of mathematics contributes to its power, because any single mathematical idea or process can accurately represent a great variety of real-world phenomena. For example, a fraction like 1/4 can convey important information on an advertisement, recipe card, gauge, sale brochure, timesheet, and so on. The abstract nature of mathematics which enables it to serve as a model from widely diverse physical phenomena may also create substantial impediments to learning. This is especially true if the teacher presents the mathematical concept in its abstract form without giving examples of its concrete representation.

Generality of concepts. Teachers often do not fully explain the general applicability of a mathematical idea or process to students, even within the limits of the students' ability. For example, teachers can explain that the method used to find the sum of the angles of a quadrilateral works equally

well for finding the sum of the angles of any polygon. Similarly, the term vertex applies to all polygons and not just to triangles, as it is sometimes introduced. Clearly, only a few of the applications of a mathematical idea should be introduced in the initial discussion of the idea. Ideally, the concept or skill should be developed in as general a setting as possible rather than artificially confined to a narrow set of conditions.

Attending to language. Teachers should be careful to use precise terminology when talking about mathematical ideas. Terms like divisor, numerator, should be used frequently such that students begin to understand and to appreciate the language of mathematics. Confusing terminology, inconsistency in expression, make it more difficult for students to learn mathematical ideas. The terms that are routinely used by mathematicians should be used in the classroom in a rich way in which students learn to talk about mathematical ideas in a practical and useful way.

Improving Development Lessons

With the passage of time, Doug Grouws and I realize that some of our assumptions about improving the development portion of the lesson were too simplistic. Part of the problem was that the initial group of teachers we worked with and the quality of mathematics supervision and resources were better than average. Having now had the opportunity to work with teachers in a variety of school districts, we appreciate more fully the advantages inherent in our initial sample. Although evidence continues to accumulate that exposure to the active teaching model is generally associated with increased student achievement, we now recognize that there is a wide range of teacher responses to the training and that teachers' knowledge of mathematics and of teaching mathematics is not evenly distributed across school districts.

We have come to believe that teachers need a great deal of assistance in

conceptualizing and conducting high-quality development lessons. Although we continue to stress that teachers need to spend more time on development, we emphasize that the amount of time spent on development will vary with the topic (e.g., some topics in calculus may call for one or more complete class periods of development), the context (early versus late in a unit), and with students' age (e.g., attention span). Structural/organizational concerns are not important; good development can be teacher or student mediated and can occur in large-group, small-group, or individual settings.

Perhaps the most important goal we have now in our current attempts to improve development lessons is to help teachers understand the criteria for a good development lesson. We are involving teachers in discussing the criteria, applying the criteria (i.e., those presented above) to videotapes of mathematics teaching, and in writing original development lessons and exchanging those lessons with other teachers.

We are trying to broaden the range of ways that teachers can convey knowledge to students and are encouraging teachers to have students engage in a wider variety of activities that will enable them to understand as well as to do mathematics. For certain teachers to change, or to achieve more lasting changes in instruction, teachers' mathematical knowledge may need to be increased or more dynamic procedures will have to be used (live observation or video analysis of one's own teaching rather than a relative more passive process, study of other's videotapes, write lessons, read those of fellow teachers). However, at present we believe that encouraging teachers to focus on development by allowing teachers at a certain grade level to meet, to plan, and to write about development (during school time), is a reasonable approach.

TABLE 1*

General Dimensions of Teachers' Communication of Differential Expectations: Selected Examples

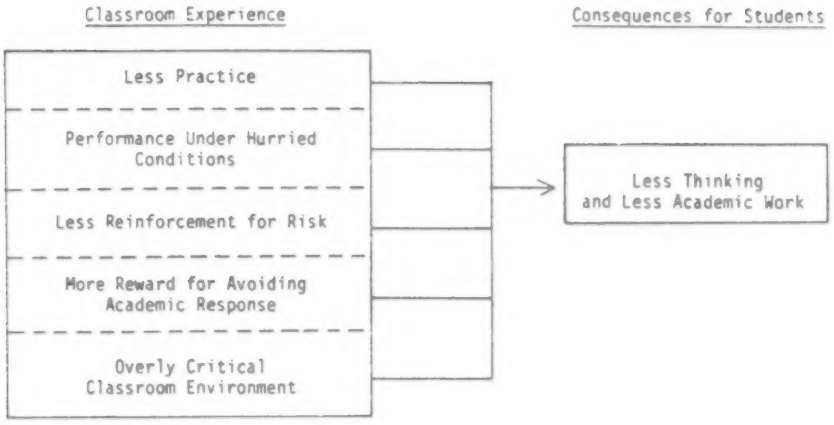
	Students Believed to be More Capable Have:	Students Believed to be Less Capable Have:
TASK ENVIRONMENT Curriculum, procedures, task definition, pacing, qualities of environment	* More variety in assignments, more complex tasks, and more conceptual work.	* Less variety (simple task structure)--less conceptualization --more practice.
	* More opportunity to perform publicly on meaningful tasks.	* Less opportunity to perform publicly, especially on meaningful tasks (supplying alternate endings to a story versus learning to pronounce a word correctly).
	* More opportunity to think.	* Less opportunity to think, analyze (since much work is aimed at practice).
	* More rapid pace.	* Less rapid pace.
GROUPING PRACTICES	* More assignments that deal with comprehension, understanding (in higher-ability groups).	* Less choice on curriculum assignments--more opportunity to work on drill-like assignments.
	* More opportunity to work with other students of high ability and motivation.	* Less opportunity to work with other students of high ability and motivation.
LOCUS OF RESPONSIBILITY FOR LEARNING	* More autonomy (more choice in assignments, fewer interruptions).	* Less autonomy (frequent teacher monitoring of work, frequent interruptions).
FEEDBACK AND EVALUATION PRACTICES	* More opportunity for self-evaluation.	* Less opportunity for self-evaluation.
MOTIVATIONAL STRATEGIES	* More honest/contingent feedback.	* Less honest/more gratuitous/less contingent feedback.
	* More new content.	* Less new content; more review.
QUALITY OF TEACHER RELATIONSHIPS	* More respect for the learner as an individual with unique interests and needs.	* Less respect for the learner as an individual with unique interests and needs.

*This table is adapted from Good, T., and Weinstein, R. (1986). Teacher expectations: A framework for exploring classrooms. In K. Zumwalt (Ed.), Improving teaching: 1986 ASCD Yearbook. Alexandria, VA.

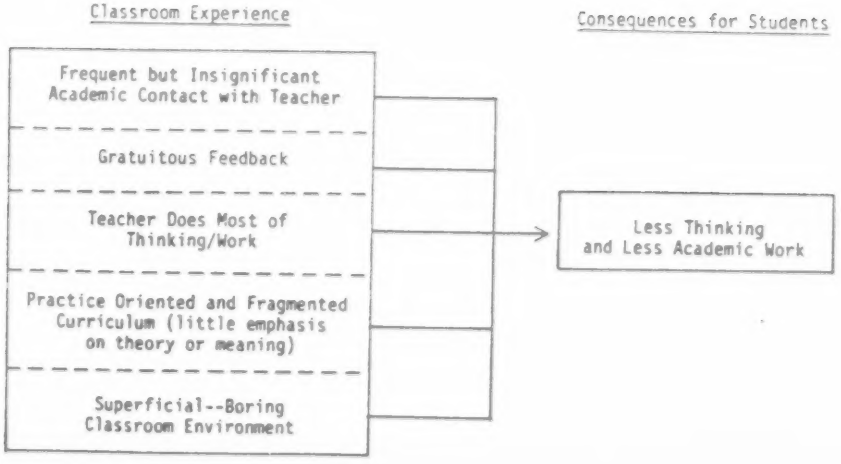
TABLE 2

Passivity Model for Low Achievers

MODEL I: Blame the Victim



MODEL II: Learner as Helpless


Classroom Experience
Consequences for Students
Frequent but Insignificant Academic Contact with Teacher
Gratuitous Feedback
Teacher Does Most of Thinking/Work
Practice Oriented and Fragmented Curriculum (little emphasis on theory or meaning)
Superficial--Boring Classroom Environment
Less Thinking and Less Academic Work

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